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Phénomènes ondulatoires dans un modèle discret de faille sismique

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Résumé

Dans cette thèse on s'intéresse à des phénomènes ondulatoires dans un modèle de faille sismique introduit par Burridge et Knopoff, et constitué d'une chaîne de patins-ressorts dans lequel des mouvements de type glissement-saccadé (*stick-slip*), caractéristiques du phénomène de tremblement de terre, sont observés numériquement.
Dans la première partie, on considère une version introduite par Carlson et Langer, avec loi de frottement de type *velocity-weakening* (adoucissement du frottement avec la vitesse de glissement). Cette loi est non lisse et multivaluée en 0. Les équations du mouvement sont alors constituées d'un système infini d'inclusions différentielles couplées. On démontre en se basant sur la méthode de Lyapounov-Schmidt, l'existence d'ondes périodiques progressives dans une limite de faible couplage entre les masses.
Dans la deuxième partie, on étudie ce modèle avec une loi de frottement de type *rate-and-state* qui prend en compte l'état de l'interface entre les deux plaques sismiques. La loi de frottement est cette fois lisse, mais dépend d'une variable d'état supplémentaire. On dérive formellement une équation de Ginzburg-Landau (GL) comme équation d'amplitude et on montre qu'il existe des petites solutions du système décrites par l'équation de GL, lorsque celui-ci se trouve au seuil de l'instabilité et sur une échelle de temps suffisamment grande.

Mots clefs : Systèmes non lisses, inclusions différentielles, bifurcations, méthode de Lypounov-Schmidt, équations d'amplitude, équation de Ginzburg-Landau.

Abstract

In this thesis, we consider a simple version of the spring-block model of Burridge-Knopoff for seismic faults, in which stick-slip instabilities have been observed numerically (phenomena corresponding to earthquakes).
In the first part, we consider the version of this model introduced by Carlson and Langer, in which the friction law is of type *velocity-weakening*. This law is nonsmooth and multivalued at zero sliding velocity. As equations of motion, we obtain an infinite system of coupled differential inclusions. We prove, using the Lyapounov-Schmidt reduction, that there exist periodic travelling waves in this system in a limit of weak coupling between the masses.
In the second part, we consider the model combined with a *rate-and-state* friction law, taking into account the *ageing* of the interface. The friction law is smooth but depends on an additive variable accounting for the state of the surface. In this part, we formally derive a Ginzburg-Landau equation as a modulation equation and prove that there exist small solutions in our system, that can be described by this equation in sufficiently large time-scale, when the system lies at the threshold of instability.

Keywords : Nonsmooth dynamical systems, differential inclusions, bifurcations, Lypounov - Schmidt reduction, modulation theory, complex Ginzburg-Landau equation.

A mon père

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Chapitre 1

Introduction Générale

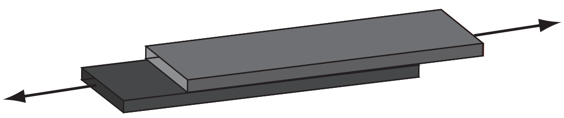
Contexte et modèle de Burridge-Knopoff

Modèle de faille sismique.

Un premier mécanisme générateur de tremblement de terre est lié à l'apparition d'une fracture dans une roche. On a donc dans ce cas ouverture d'une fissure puis propagation dans une roche sous contrainte. Un autre mécanisme, plus fréquent, résulte d'un glissement de deux blocs au niveau d'une faille pré-existante. Dans ce cas, il s'agit d'un phénomène plus de frottement et non de fracture et ce sont alors les mécanismes de frottement qui permettent d'expliquer l'évolution de la contrainte. Dès qu'une faille est formée, toute augmentation de contrainte se traduira plus fréquemment par le glissement le long de cette interface plutôt que par la fracturation de roches intactes.

Une manière simple de modéliser une faille sismique est de considérer deux plaques élastiques comprimées l'une contre l'autre et contraintes de se déplacer en direction opposée le long de leur ligne de contact. Les deux plaques restent en équilibre tant que la contrainte de cisaillement au niveau de la faille est assez faible. Quand cette contrainte dépasse un seuil critique, il se produit un glissement des plaques, caractéristique du phénomène de tremblement de terre. Les cycles sismiques correspondent alors à des oscillations de type glissement saccadé (*stick-slip*) entre des états d'équilibre et de glissement (cf. [Ren98]).

La modélisation des failles sismiques met en jeu un certain nombre de modèles simplifiés qui prennent en compte les degrés de liberté essentiels pour décrire la dynamique de l'interface. Bien qu'ils fassent appel à différentes hypothèses simplificatrices (propa-



gation en milieu unidimensionnel, élasticité à portée limitée, homogénéité spatiale), leur étude mathématique est délicate car ces modèles mettent en jeu des lois de friction non linéaires qui peuvent être également non régulières et non univoques comme dans le cas du frottement de Coulomb, conduisant pour les équations du mouvements, à des inclusions différentielles.

Le modèle que nous considérons ici est un modèle de patin-ressort aujourd'hui standard, introduit par Burridge et Knopoff dans les années 1960 [BK67]. Il consiste en une chaîne de blocs de masse m, couplés par des ressorts de raideur k_c, et liés à une surface inférieure rugueuse qui se déplace à une vitesse v non nulle (voir figure 1.1).

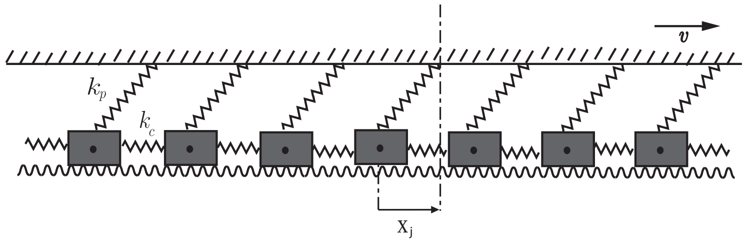


FIGURE 1.1 : Modèle de Burrige-Knopoff, d'après Carlson et Langer (1989)

La chaîne de blocs représente une discrétisation de l'un des côtés de la faille, et la ligne de faille correspond à la surface de contact entre les blocs et le support rugueux.

Lois de frottement.

La loi de frottement décrit l'évolution du coefficient de frottement μ, défini comme le rapport de la force tangentielle F_T et de la force normale N exercées sur l'interface frottante :

$$\mu = \frac{F_T}{N}.$$

Cette évolution est fonction des paramètres physiques du contact : distance de glissement, vitesse de glissement, état de l'interface... La loi de frottement la plus classique est la loi de Coulomb, qui décrit le frottement comme un phénomène à un seuil. Au repos (vitesse de glissement $V = 0$), les propriétés de contact n'indiquent qu'une borne supérieure : $\mu < \mu_s$, où μ_s est le *coefficient de frottement statique*. En situation dynamique ($V > 0$), la loi prédit que le coefficient de frottement est constant $\mu = \mu_d$ où μ_d est le *coefficient de frottement dynamique*. De plus, en général on a $\mu_d < \mu_s$, ce qui traduit un adoucissement instantané du frottement lors de l'initialisation du glissement. Cette loi, bien qu'idéalisée, rend bien compte des phénomènes de frottement observés en laboratoire. On peut cependant l'affiner de plusieurs manières (voir [Ren98, Rui83]).

- En considérant que l'adoucissement du frottement ne se fait plus instantanément, mais sur une quantité de glissement fini : on aboutit alors à la classe des lois de frottement de type SWF (dites "**slip-weakening**"). En notant x_j la déviation du j-ième bloc par rapport à sa position d'équilibre. Les équations régissant le système sont celles provenant de la dynamique Newtonienne et dans ce cas sont du type

$$m\ddot{x}_j = k_c(x_{j+1} - 2x_j + x_{j-1}) - k_p x_j - F(x_j), \quad j \in \mathbb{Z}, \tag{1.1}$$

où l'on a noté F la loi de frottement.

- En prenant en compte lors de la phase de glissement les variations de μ_d en fonction de la vitesse de glissement : on aboutit à des lois de type "**velocity-weakening**", notamment utilisée par Carlson et Langer [CL89a, CL89b] ainsi que Schmittbuhl et al. [SVR93]. En d'autres termes, on considère ici que le frottement solide-solide diminue avec la vitesse. Dans ce cas, les équations du mouvement sont du type :

$$m\ddot{x}_j \in k_c(x_{j+1} - 2x_j + x_{j-1}) - k_p x_j - F(v + \dot{x}_j), \quad j \in \mathbb{Z}, \tag{1.2}$$

où $v+\dot{x}_j$ correspond à la vitesse de glissement. Il s'agit ici d'inclusions différentielles à cause du phénomène de seuil (de type Coulomb) lorsque le système est au repos, traduit par une multivaluation en 0 de la loi de frottement (voir Figure 1.2). Dans la partie I de cette thèse, nous considèrerons ce type de frottement et notamment la loi introduite par Carlson et Langer : il s'agit d'une loi spatialement uniforme et ne dépendant pas de l'état de l'interface.
L'objet de cette partie sera de prouver l'existence d'ondes périodiques progressives dans ce système non lisse.

- Beaucoup d'autres facteurs influent sur la valeur du coefficient de frottement. Notamment les paramètres physiques de l'interface, l'âge des contacts ou encore l'histoire du glissement. Le modèle de frottement RSF, "**rate-and-state**" a été élaboré à partir d'expériences en laboratoire, notamment avec des travaux de Dieterich [Die79], Ruina [Rui83], Marone [Mar98]. Ces lois prennent en compte des petites dépendances du coefficient de frottement avec la vitesse de glissement ainsi qu'une collection de *variables d'état* θ qui décrivent l'*état de l'interface*. Les équations du mouvement sont cette fois couplées avec une équation d'évolution pour la variable d'état :

$$\begin{cases} m\ddot{x}_j = k_c(x_{j+1} - 2x_j + x_{j-1}) - k_p x_j - F(v + \dot{x}_j, \theta_j), \quad j \in \mathbb{Z}, \\ \dot{\theta}_j = G(\theta_j, v + \dot{x}_j), \quad j \in \mathbb{Z}. \end{cases}$$

Dans la partie II de cette thèse, nous étudierons le modèle de Burrigde-Knopoff avec une loi simple de loi RSF introduite par Dieterich et Ruina [Die79, Rui83], et qui dépend effectivement de l'état de l'interface via une unique variable d'état.
Contrairement au système avec la loi velocity-weakening de Carlson et Langer, nous obtenons un système lisse et donc nous pouvons utiliser les outils classiques d'analyse lisse. En l'occurrence, notre objectif dans cette partie sera de dériver une équation d'amplitude de Ginzburg-Landau qui décrit effectivement la dynamique des petites solutions du système, dans un certain régime de paramètres.

Sous certaines conditions, les lois RSF généralisent les lois de Coulomb ainsi que les lois d'adoucissement en glissement. Ce modèle non linéaire tire son succès d'une très bonne description du glissement pour de nombreux matériaux testés en laboratoire et permet d'expliquer de nombreux phénomènes sismologiques.

Partie I : Ondes progressives dans le modèle de Burridge-Knopoff, version de Carlson et Langer

Description du modèle

Dans la première partie de cette thèse, nous considérons une version de ce modèle due à Carslon et Langer [CL89a, CL89b], dans laquelle la chaîne est spatialement homogène et le coefficient de frottement dynamique est une fonction non linéaire et décroissante de la vitesse de glissement. Ce modèle met en jeu les phénomènes de chargement mécanique, stockage de l'énergie élastique et glissement saccadé que l'on rencontre au niveau des failles sismiques. Il reproduit notamment dans une certaine plage de magnitude, la loi de Gutenberg-Richter, selon laquelle la fréquence des évènements sismiques décroît linéairement avec leur magnitude [CL89a, CL89b]. Dans leur modèle, les forces de frottement auxquelles sont soumises les plaques sont modélisées par une loi de frottement non linéaire de type Coulomb. Ce modèle est déterministe et sans variation spatiale des paramètres : on ne prend pas en compte d'éventuelles inhomogénéités spatiales.
On note x_j la déviation du j-ième bloc par rapport à sa position d'équilibre. Les équations régissant le système sont celles provenant de la dynamique Newtonienne :

$$m\ddot{x}_j = k_c(x_{j+1} - 2x_j + x_{j-1}) - k_p x_j - N(v + \dot{x}_j), \quad j \in \mathbb{Z}, \tag{1.3}$$

avec la loi de frottement N donnée par (voir Figure 1.2) :

$$N(y) = N_s \, \Psi(y/v_1), \;\; \Psi(y) = \frac{\mathrm{sgn}(y)}{1 + |y|}, \tag{1.4}$$

où $\mathrm{sgn}(y)$ est mulivaluée en 0, $|\mathrm{sgn}(0)| < 1$, et avec v_1 une vitesse caractéristique qui fixe l'échelle de la loi de frottement. On considèrera une chaîne infinie de masses afin d'étudier les solutions propagatives de cette équation ($j \in \mathbb{Z}$). Afin de minimiser le nombre de paramètres, on adimensionne ces équations. On obtient :

$$\boxed{\ddot{u}_j + u_j = \ell^2\,(u_{j+1} - 2u_j + u_{j-1}) - F_0\Psi(V + \dot{u}_j),} \tag{1.5}$$

où $\ell = k_c/k_p$ est le *paramètre de couplage*, V et F_0 sont sans dimension.

D'après les simulations de Carlson et Langer, la dynamique de ce système est très complexe. Quand les conditions initiales sont spatialement uniformes, le système est soumis à un mouvement de type saccadé (tout comme il se comporterait dans le cas d'un unique bloc). Mais cet état est instable : en introduisant une petite inhomogénéité dans la condition initiale, les blocs glissent à peu près au même moment, mais les irrégularités sont

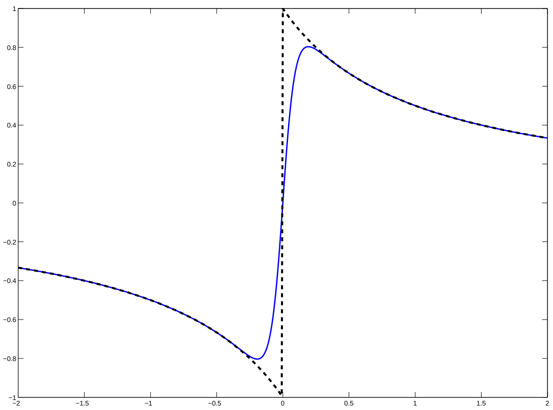


FIGURE 1.2 : Loi de frottement de Carlson et Langer de type velocity-weakening (courbe noire en pointillés) et loi de frottement régularisée (courbe bleu continue)

amplifiées pendant le glissement et l'état atteint au repos est très irrégulier. D'autre part, ils ont constaté que beaucoup des évènements apparaissant dans ce système n'impliquent qu'un petit nombre de blocs. En d'autres termes, ces évènements sont *périodiques* avec un motif *localisé* (voir Figure 1.3).

Question mathématique

Ce modèle a été très étudié numériquement. En particulier, Schmittbuhl et al. [SVR93] l'ont simulé avec un nombre fini de blocs N et avec des conditions aux limites périodiques. Ils ont mis en évidence le rôle du paramètre $\theta = \nu N$ (où l'on a noté $\nu = \frac{vk_p}{\omega_p N_s}$ et $\omega_p^2 = \frac{k_p}{m}$). Pour des valeurs assez grandes de θ, Schmittbuhl et al ont montré que la propagation de zones de glissement très localisées est possible dans le système. Il s'agit d'ondes progressives périodiques dont la vitesse est sélectionnée par les paramètres du modèle. En revanche, pour θ petit, la dynamique est dominée par des oscillations collectives de type glissement saccadé, mais des ondes localisées sont tout de même observées sur des intervalles de temps limité (voir Figure 1.4).
L'existence de ces ondes progressives observées numériquement pour ce système était jusqu'à présent un problème ouvert du point de vue théorique. Le premier problème qu'on se pose est d'apporter une preuve mathématique à ces résultats. Le premier problème étudié dans cette thèse est donc le suivant :

Problème I. *Peut-on prouver un théorème d'existence d'ondes périodiques progressives localisées confirmant les résultats numériques obtenus par Schmitthbuhl et al. dans [SVR93] ?*

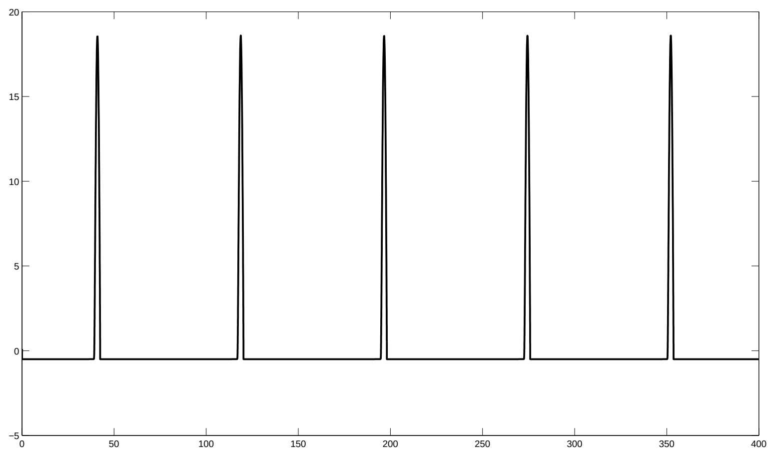


FIGURE 1.3 : Graphe de $\dot{u}(\xi)$ (onde périodique à motif localisé).

Notons que des études dans la limite du continu ont été menées avec ce modèle de Burridge-Knopoff. Notamment, dans [Mur99], où l'existence d'ondes de choc avec une loi de type Coulomb est abordée. Mais dans cette thèse nous restons dans un cadre discret.

Difficultés et outils employés

Difficultés La théorie des oscillations non linéaires dans des systèmes non lisses de petites dimensions est maintenant bien développée (voir par exemple [BBCK08]) mais au vu du système (1.5) nous avons deux principales difficultés : le système infini d'EDO couplées ($j \in \mathbb{Z}$) et le caractère multivalué en 0 de la loi de frottement qui induit des inclusions différentielles.
Nous recherchons des ondes progressives périodiques. On cherche donc des solutions sous la forme $u_j(t) = u(\xi) = u(j + t/\tau)$ se propageant à vitesse constante $1/\tau$ et on injecte cet ansatz dans le système. On obtient alors :

$$\frac{\ddot{u}}{\tau^2} + u \in \ell^2 \left(u(\xi+1) - 2u(\xi) + u(\xi-1)\right) - F\left(V + \frac{\dot{u}}{\tau}\right). \tag{1.6}$$

Ainsi on perd la difficulté liée au système infini, mais il subsiste toujours deux difficultés à surmonter pour répondre au problème I :

- l'**inclusion différentielle** (à cause de la multivaluation en 0 de la loi de frottement)
- le **terme d'avance/retard** apparu en contrepartie du système infini d'EDO couplées

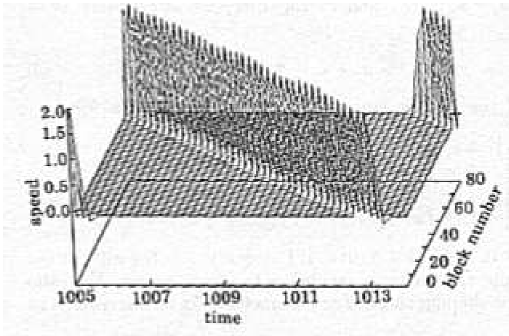


FIGURE 1.4 : Propagation d'une zone de glissement localisée dans le modèle de Carlson et Langer avec conditions aux limites périodiques (Schmittbuhl et al, 93).

L'inclusion (1.6) dépend en outre de 2 paramètres réels, V et ℓ et de la vitesse de l'onde $\frac{1}{\tau}$ qui est considérée comme une inconnue du problème.

Outils On commence par faire abstraction de la première difficulté en regardant ce problème dans lequel on remplace la loi de frottement F non lisse par une *loi régularisée* F_ε (voir 1.2) univoque. On obtient alors une équation différentielle d'ordre 2 régulière dans lequel subsiste le terme d'avance/retard :

$$\frac{\ddot{u}}{\tau^2} + u = \ell^2 \left(u(\xi+1) - 2u(\xi) + u(\xi-1) \right) - F_\varepsilon \left(V + \frac{\dot{u}}{\tau} \right). \tag{1.7}$$

Le second membre étant régulier, on peut maintenant utiliser les méthodes classiques d'analyse lisse pour répondre à la question. La notion de limite anti-continue a été introduite par Aubry et MacKay (voir [Aub97, mKA94]) pour trouver des breathers dans le cadre de réseaux hamiltoniens, puis étendue dans le cadre dissipatif par Sepulchre et MacKay (voir [SmK97]). La méthode consiste à montrer qu'on a continuation des solutions qui existent dans le cas plus simple où $\ell = 0$. Nous utiliserons pour cela une approche perturbative, la méthode de Lyapounov-Schmidt, qui est un raffinement du Théorème des Fonctions Implicites. On rappelle qu'on cherche une solution périodique de (1.7). Pour cela :

1. On cherche cette solution périodique sans le terme avance/retard (donc à couplage ℓ nul).

2. On montre la persistance via la méthode de Lyapounov-Schmidt de cette solution périodique quand on rajoute le terme de couplage, mais dans une limite de faible couplage.

Une fois bien compris ce problème lisse, on s'intéresse au problème initial non lisse (1.6), plus délicat à cause de la double difficulté évoquée précédemment. L'approche par la méthode de Lyapounov-Schmidt n'est plus applicable telle qu'elle à cause de la multivaluation en 0. On doit alors affiner cette stratégie en décrivant plus précisément la forme de la solution périodique recherchée.

Travail réalisé

Cette partie est découpée en deux chapitres. Le premier concerne le problème lisse (avec loi de frottement régularisée) et le second le problème non lisse.

Dans les deux cas, nous montrons que le problème sans avance/retard ($\ell = 0$) possède une solution périodique en utilisant le Théorème de Poincaré-Bendixson (dont une version non lisse pour le deuxième cas). On peut également montrer dans le cas lisse que cette orbite nait d'une bifurcation de Hopf et qu'elle est stable.

Puis nous démontrons dans chacun des cas un théorème d'existence du type :

Théorème 1.1

Soit u_0 une orbite périodique du système (1.6) *pour $\ell = 0$ et $\tau = \tau_0$. Alors sous une condition de non dégénérescence, pour tout ℓ dans un voisinage de* 0*, il existe une orbite périodique proche de u_0 en graphe, de même période que u_0, de vitesse inverse τ proche de τ_0, qui est solution du système faiblement couplé* (1.6)*.*

Pour les énoncés plus précis, voir les Théorèmes 3.3 (page 37) et 2.7 (page 28).

Dans le cas lisse, la condition de non dégénérescence se traduit par une condition sur les multiplicateurs de Floquet : 1 est multiplicateur de Floquet simple du système linéarisé autour de la solution d'équilibre.

Idées de la preuve

- Dans le cas lisse, on cherche u sous la forme $u = u_0 + u_1$, où u_1 est une petite perturbation, se propageant à vitesse $\tau \approx \tau_0$, et de même période T_0 que u_0. On a donc deux inconnues, u_1 et τ. On réécrit le problème comme une équation implicite en u_1, τ et ℓ. La différentielle n'étant pas inversible en $(u_0, \tau_0, 0)$, on met en oeuvre la méthode de Lyapounov-Schmidt en projetant sur des espaces adaptés. La première projection nous permet d'écrire par le Théorème des Fonctions Implicites u_1 en fonction de τ et ℓ. La deuxième projection nous donne une équation par laquelle, moyennant notre hypothèse de non dégénérescence, on peut extraire τ en fonction de ℓ.

- Dans le cas non lisse, notre équation implicite en u_1, τ, ℓ, devient une *inclusion* implicite, et donc il est nécessaire d'affiner la stratégie. On contourne ce problème de la manière suivante : on impose à vitesse nulle (qui correspond à la période de multivaluation), une expression explicite à notre solution. A vitesse non nulle, on cherche notre solution comme une petite perturbation de l'orbite périodique u_0. Mais sur cet intervalle de temps, l'inclusion devient une *équation.* Il est donc possible de réutiliser la méthode de Lyapounov-Schmidt. Autrement dit, on impose un mouvement de type stick-slip. Ce découpage nécessite bien entendu des conditions de raccord supplémentaires.

Remarque 1.1.

- *Ces méthodes perturbatives trouvent leur limite dans le faible couplage ℓ.*
- *Ici on ne capture qu'une solution périodique. Et donc contrairement au problème de la partie II, nous ne décrivons par la dynamique au voisinage d'une solution de base. En revanche, puisque les solutions du problème découplé sont obtenues par une méthode non locale (Théorème de Poincaré-Bendixson), nous n'avons pas de limitation sur la taille de l'orbite périodique.*
- *Des simulations numériques de ces orbites avec matlab nous permettent de constater que ces solutions sont effectivement localisées.*
- *Nous n'avons pas de résultat de stabilité concernant les solutions capturées. C'est un prolongement possible de ce travail. Dans un premier temps, nous pourrions calculer numériquement ces solutions en utilisant des méthodes adaptées aux systèmes non lisses (voir par exemple [AB08]), et calculer les coefficients de Floquet numériquement.*

Partie II : Justification d'équations d'enveloppe dans le modèle de Burridge-Knopoff, avec loi rate-and-state de Dieterich-Ruina

Description du modèle

L'idée des lois RSF est la suivante. On considère qu'à un instant donné, la surface a un *état* e (*state*) et que la contrainte de frottement τ, dépend de la vitesse de glissement V, de la contrainte normale σ et de e : $\tau = F(\sigma, V, e)$. A tout point de la surface, la variation de cet état (*rate*) n'est supposé dépendre que de l'état à ce même point, de σ et de V : $\frac{de}{dt} = G(\sigma, V, e)$. Cet état peut changer avec des paramètres extérieurs, comme la température, la pression... Il est représenté par une collection de *variables d'état* θ_i. En considérant que la contrainte normale σ est constante et que la contrainte τ lui est proportionnelle, on obtient :

$$\begin{aligned} \tau &= \sigma F(V, \theta_1, \theta_2, \ldots), \\ \frac{d\theta_i}{dt} &= G_i(V, \theta_1, \theta_2, \ldots). \end{aligned}$$

La version la plus standard de loi RSF est la loi de Dieterich-Ruina [Die79, Rui83] déterminée sur de nombreuses observations expérimentales :

$$\mu = \mu_0 + a \ln\left(\frac{V}{V_0}\right) + b \ln\left(\frac{V_0\,\theta}{d_c}\right),$$

où μ_0 est le coefficient de frottement statique à $V = 0$, V est la vitesse de glissement, V_0 la vitesse initiale, d_c est la distance de glissement critique, *i.e.* la distance moyenne en cas de changement de la vitesse de glissement V pour atteindre un nouvel état d'équilibre, a et b sont des constantes sans dimension dépendant des matériaux et déterminées expérimentalement. Cette loi a été déterminée en étudiant expérimentalement l'influence des sauts de vitesse et de l'arrêt du glissement sur le coefficient de frottement μ. De plus, elle ne dépend que d'une unique variable d'état θ qui suit la loi d'évolution suivante :

$$\frac{d\theta}{dt} = 1 - \frac{V\theta}{d_c}.$$

La dépendance logarithmique en la vitesse de glissement V n'est pas adaptée pour les très grandes ou les très petites vitesses de glissement. De plus, on peut également affiner cette loi en rajoutant d'autres variables d'état. Des travaux de Baumberger et Caroli [BBC99] ont permis de comprendre l'origine physique de la variable d'état du modèle de Dieterich-Ruina : θ représente l'âge moyen des contacts le long de l'interface. Ainsi la dépendance en θ de la loi de frottement est due à un processus de vieillissement des aspérités de contact (dû au fluage sous contrainte normale).
Notons enfin que l'influence de l'élasticité évite d'avoir recours à une version régularisée de la loi de Dieterich-Ruina lors des phases d'arrêt : grâce à l'élasticité, la vitesse conserve à tout instant des valeurs finies au cours des phases d'arrêt et la loi continue d'être applicable.

Les équations du mouvement pour le j-ième bloc dans le cas du modèle de Burridge-Knopoff combiné à une loi de type RSF que nous étudierons sont celles de l'article de [OK07] :

$$\begin{cases} m\ddot{x}_j = k_p(vt - x_j) + k_c\,(x_{j+1} - 2x_j + x_{j-1}) - \Phi(\dot{x}_j, \theta_j), \\ \dfrac{d\theta_j}{dt} = 1 - \dfrac{\dot{x}_j\theta_j}{d_c}, \end{cases}$$

avec

$$\Phi(\dot{x}_j, \theta_j) = \sigma\left\{c + a\ln\left(1 + \frac{\dot{x}_j}{v^*}\right) + b\ln\left(\frac{v^*\theta_j}{d_c}\right)\right\}. \tag{1.8}$$

Les paramètres a, b et c sont constants, d_c est une distance caractéristique de glissement, σ est la charge normale (constante) , v^* une vitesse de référence et v est la vitesse (constante) de la plaque supérieure.
Les équations adimensionnées du modèle sont (dans le repère en translation à vitesse v) :

$$\boxed{\begin{cases} \ddot{v}_j + v_j = \ell^2\,(v_{j+1} - 2v_j + v_{j-1}) - \{c + a\ln(1 + v + \dot{v}_j) + b\ln\theta_j\}\,, \\ \qquad \dot{\theta}_j = 1 - (\dot{v}_j + v)\theta_j. \end{cases}} \tag{1.9}$$

Question mathématique

On se pose maintenant une question de nature complètement différente et qui engage des outils qui diffèrent de ceux utilisés pour le premier problème. La question est la suivante :

Problème II. *Peut-on trouver un modèle plus simple que le système* (1.9)*, et qui permette de décrire la dynamique des solutions de* (1.9) *qui sont sous la forme d'ondes modulées de petites amplitudes ?*

Nous pourrions en réalité démontrer des résultats similaires dans ce cadre à ceux obtenus dans le problème I. Inversement, les résultats de la partie II sont aussi démontrables dans le cadre de la loi de Carlson et Langer régularisée, mais il est intéressant d'étudier le système avec ce type de lois, au vu du succès qu'elles ont pour décrire ce frottement. Ce problème a été abordé de façon heuristique par Hähner et Drossinos [HD98, HD99] pour une variante du problème (1.9), continu en espace et comportant une loi rate-and-state d'un type différent. Les auteurs décrivent de façon heuristique une équation de Ginzburg-Landau pour approcher la dynamique du système au voisinage d'un état de glissement uniforme. Dans la partie II de cette thèse, on démontre un théorème d'approximation des solutions de (1.9) par des solutions d'une équation de Ginzburg-Landau.

Difficultés et outils mathématiques

Nous répondons à cette question en utilisant la théorie des équations d'amplitude dont nous décrivons les principales idées ici.

Etude de bifurcation Nos systèmes (1.5) et (1.9) dépendent d'un paramètre (dans $\mathbb{R}$ pour la loi de Carlson et Langer, et dans $\mathbb{R}^4$ pour la loi rate-and-state). Ces problèmes possèdent un état stationnaire. Dès lors, on s'intéresse à l'étude de bifurcation, *i.e.* au changement de stabilité de cet état, qui s'accompagne de l'apparition de nouvelles solutions. Historiquement, le premier outil utilisé pour s'attaquer à ces questions est la méthode de Lyapounov-Schmidt, qui est une méthode perturbative comme on l'a vu et qui permet notamment de prouver la persistance de solution périodiques en rajoutant des "termes petits". Cette méthode permet donc de trouver des solutions particulières (orbites périodiques, homoclines,. . .) classe de solutions par classe de solutions.
D'autre part, une deuxième classe de méthodes d'analyse de bifurcation est le Théorème de la Variété Centrale (cf [HI10], Théorème 3.3 p.46). Contrairement aux méthodes précédentes, celui-ci nous permet de décrire la dynamique des solutions proches de l'origine avec certaines conditions spectrales (voir [IJ05, JS08, IK00] pour des applications dans des réseaux).

Théorème de la Variété Centrale et parallèle avec la théorie des équations d'amplitude Ce théorème dit que pour un problème d'évolution dans un espace de Banach, au voisinage d'un équilibre, sous certaines hypothèses spectrales, il existe une variété locale de dimension finie, invariante par le flot et telle que cette variété contient toutes les solutions qui restent dans un voisinage de l'équilibre en temps $t \in \mathbb{R}$. L'équation vérifiée par les solutions sur cette variété (dite équation réduite) décrit en ce sens la dynamique locale.
De plus, si le spectre ne contient aucun élément à partie réelle positive, alors cette variété est attractive au sens suivant : toute petite solution possède une ombre sur la variété et converge de manière exponentielle vers celle-ci.

Toutefois les hypothèses spectrales de ce théorème ne sont pas vérifiées pour tous les problèmes physiques, par exemple typiquement parce que le spectre est continu.

La théorie des équations de modulation peut alors être vue comme une alternative lorsque les hypothèses spectrales du Théorème de la Variété Centrale ne sont pas satisfaites. Elle a pour but également de décrire la dynamique des petites solutions d'un problème lorsque le système est au seuil de l'instabilité dans le sens suivant : les petites solutions du système qui sont proches initialement d'une famille d'ondes périodiques progressives planes monochromatiques (OPPM) modulées dont l'amplitude est solution d'une équation dite d'*amplitude* ou d'*enveloppe*, restent proche de cette OPPM modulée sur une échelle de temps grande. De plus, cette famille d'OPPM modulées est un ensemble attractif pour les solutions du système (localement autour de 0).

En ce sens, l'équation d'amplitude que l'on cherche à obtenir est le pendant de l'équation réduite de la variété centrale. Toutefois cette approche diffère de manière importante avec celle de la variété centrale par le fait que les approximations de solutions se font sur des intervalles de temps finis. Il est toutefois possible dans certains cas de décrire globalement les solutions du système initial par l'équation d'amplitude en considérant des pseudo-orbites de celle-ci (voir plus loin).

Grandes lignes de la théorie de la modulation Cette méthode d'approximation de solution par des ondes modulées était déjà bien connue des physiciens dans les années 60. Elle a été développée pour décrire les modulations en temps et espace d'ondes planes progressives monochromatiques dans un système quand un paramètre atteint une valeur critique. Parmi les premiers exemples d'approximation dans des problèmes physiques, citons le problème de convection de Rayleigh-Bénard (voir [NW69], [Seg69]) lié au phénomène de cellules ou rouleaux de convection apparaissant quand on chauffe un liquide avec une source extérieure. Le paramètre de bifurcation est le nombre de Rayleigh R. Dans [Seg69], l'auteur montre qu'une variation lente d'espace permet de décrire les solutions du problème avec bords. L'amplitude des rouleaux doit vérifier l'équation d'amplitude quand on rajoute les bords.

La base de la théorie est donnée par l'idée suivante. Considérons un système (scalaire pour faire simple) $\partial_t u = L(\mu, \partial_x)u + N(\mu, \partial_x, u)$ admettant 0 comme équilibre et tel que pour $\mu < 0$, le système est stable et devient instable pour $\mu > 0$ par le biais de toute une bande de modes de Fourier. Alors, si $\lambda(\mu, k)$ est une valeur propre de $L(\mu, ik)$, la famille d'ondes $e^{ikx+\lambda t}$ est solution du problème linéaire. Si $\mu = 0$, et que k_0 est associé à une valeur propre imaginaire pure, alors la théorie linéaire prédit que tous ces modes sont amortis sauf le mode critique k_0. Lorsque $\mu > 0$, si on ne prend pas en compte les termes non linéaires, l'analyse linéaire prédit que les modes pour k proche de k_c croissent exponentiellement avec le temps. L'équation de modulation est alors formellement dérivée pour décrire l'évolution non linéaire des modes linéairement instables. En notant $u_A(t, x) = \varepsilon A(\tau, \xi)E(t, x)$, l'approximation construite à partir de l'OPPM $E(x, t) = e^{ik_0 x + i\omega t}$, modulée avec une amplitude A lente en espace et en temps (τ, ξ sont des variables lentes de temps et d'espace), on injecte cet ansatz multi-échelle dans l'équation initiale. Puis on égalise les puissances de $\varepsilon^j E^n$ à 0. On obtient par ce processus une équation que doit satisfaire l'amplitude A de l'approximation u_A comme condition de compatibilité lors de l'égalisation de $\varepsilon^3 E^1$. Des exemples classiques d'équation d'amplitude sont Schrödinger non linéaire [GM04, GM06], Ginzburg-Landau [KSM92, Sch94], ou encore l'équation de Cahn-Hilliard [Sch99], Schrödinger non linéaire discret [PS10, PSmK08], etc ...

On trouve aujourd'hui dans la littérature beaucoup d'approches mathématiques de la théorie. A propos du formalisme général de la théorie avec équation d'amplitude de type Ginzburg-Landau, on peut citer [Eck91, VHa91] et le papier de revue [Mie02]. Les exemples d'application mathématique de la théorie sont riches. Une première justification mathématique de problème de Rayleigh-Bénard est donnée par Schneider dans [Sch94a]. L'équation de Swift-Hohenberg (scalaire) est traitée dans [CE90] et repris dans [KSM92] avec une justification simple dans le cas où la non linéarité est cubique. Pour une description générale de la théorie appliquées à des domaines cylindriques non bornés, citons [Sch01]. Dans un cadre discret comme le notre, on peut se référer à l'article général de Giannoulis, Hermmann et Mielke [GHM06], à [GM04] pour le problème de FPU avec non linéarité quadratique, ainsi qu'à [GM06] pour le prolongement de leur étude dans le cas avec une non linéarité quadratique. Enfin, pour des questions similaires concernant des problèmes d'optique non linéaire, on pourra voir [JMR93a, JMR93b, JMR99, Col02, CL04, Lan11].

La théorie des équations d'amplitude se déroule donc en deux étapes :

- Etape 1 : dérivation formelle de l'équation d'amplitude
- Etape 2 : validité de l'équation d'amplitude (ou justification)

On a vu que la première étape consiste à injecter un ansatz d'onde modulée dans le système de départ pour obtenir l'équation d'amplitude. La deuxième étape consiste à justifier le fait que cette équation décrit effectivement la dynamique du système dans un sens que nous allons préciser. Le premier point est purement une étape de calcul formel. Le deuxième point est une question plus difficile. Et le contre-exemple traité dans [Sch95] montre bien l'importance de cette étape car l'équation de modulation obtenue formellement ne décrit pas toujours correctement les solutions du problème.

Une fois l'équation d'amplitude dérivée, on peut se poser les questions suivantes :

1. Quelles informations obtient-on pour le problème initial en étudiant l'équation d'amplitude ?

2. Les solutions A de l'équation d'amplitude génèrent-elles via u_A une bonne approximation de solutions du problème initial ? Et sur quel échelle de temps ?

La justification comporte ainsi trois sous-problèmes :

- **La propriété d'approximation**. Il s'agit d'estimer l'erreur entre l'approximation u_A et les solutions du problème initial avec condition initiale proche de $u_A(0)$. Dans le cas où la non linéarité commence par des termes cubiques, le problème est plus simple que dans le cas d'une non linéarité quadratique. En effet dans ce cas, une estimée de semi-groupe ainsi qu'un argument de type Gronwall permet de conclure. Le cas cubique est traité par exemple pour l'équation de Swift-Hohenberg dans [KSM92] ainsi que dans le cas discret dans [GM04]. Pour une non linéarité quadratique, cet argument tombe en défaut. Dans ce cas, si $k_0 \neq 0$, l'idée est de remarquer que l'interaction quadratique des modes de Fourier critiques $\pm k_0$ ne donne pas des modes critiques, ce qui n'est plus vrai si $k_0 = 0$. Or, les solutions OPPM modulées ont des transformées de Fourier qui sont concentrées autour des modes $\pm k_0$. Avec les interactions quadratiques on ne va générer que des modes non critiques, donc exponentiellement amortis. L'outil principal pour traiter ce cas est alors de séparer les modes critiques des autres modes en appliquant un filtre. Dans le cas où $k_0 = 0$ on commence par transformer le système avec la théorie des formes normales en un système dans lequel l'interaction des modes critiques ne donne plus de modes critiques (voir par exemple [Sch98, GM06]). L'outil principal de la justification qui suit est alors l'utilisation de filtres qui permettent de séparer les modes critiques du reste du spectre. Ces filtres (qui ne sont pas des projections à cause de la continuité du spectre) sont construits par transformée de Fourier et en faisant un cut-off dans l'espace de Fourier autour des modes critiques (voir [Mie02]).

- **Attractivité de l'ensemble des OPPM modulées**. Dans le problème précédent, on obtient un résultat d'approximation de solutions uniquement lorsque celles-ci ont initialement une forme OPPM modulée. Ici, l'objectif est de montrer qu'en temps fini, toutes les petites solutions du problème développe cette structure et qu'elles sont donc décrites par l'équation d'amplitude. Autrement dit, on souhaite montrer que l'ensemble de ces OPPM modulées attire toutes les solutions dont les conditions initiales sont dans un voisinage de 0. On pourra voir à ce propos l'article de Eckhaus [Eck93]. On obtient ainsi des résultats du type du théorème 5.1 de [Sch98].

- **Existence globale.** Dans le cas où l'équation d'amplitude admet des solutions globales, on peut combiner les propriétés d'attraction et d'approximation pour construire des approximations globales des petites solutions du systèmes à l'aide de pseudo-orbites (voir définition 6.3 de [Mie02]).

Travail réalisé

La première partie de ma thèse est divisée en trois chapitres. Le premier chapitre a pour objet l'analyse spectrale de notre problème, le deuxième chapitre la dérivation formelle et le troisième chapitre aborde la justification de l'équation modèle.

Chapitre 6 : Existence d'une bifurcation de Hopf "étendue". Le système dépend d'un paramètre dans $\mathbb{R}^4$. Le spectre de l'opérateur linéaire est continu paramétré par le nombre d'onde (que l'on note q). Ainsi, les hypothèses du Théorème de la Variété Centrale ne sont pas satisfaites.
On montre qu'il existe une variété critique dans $\mathbb{R}^4$ qui, lorsqu'on la traverse, induit un changement de stabilité du système. Et à la traversée de cette variété, tout une bande de modes devient instable (voir la Proposition 6.5, page 85). Le mode $q_0 = 0$ est par ailleurs le premier à devenir instable.
De plus, le spectre est composé de trois courbes lisses et nous rentrons dans le cadre spectral décrit par Mielke pour dériver une équation de Ginzburg-Landau ([Mie02]).

Chapitre 7 : Dérivation d'une équation de Ginzburg-Landau complexe. Comme prédit par le cadre spectral, nous dérivons l'équation d'amplitude suivante :

$$\partial_\tau A = (c + id)A + (\nu + i\alpha)\partial_{\xi\xi} A + (a + ib)|A|^2 A,$$

où $A(\tau, \xi) \in \mathbb{C}$ est l'amplitude de l'OPPM modulée. Le résultat précis est donné par le Théorème 7.2.

Chapitre 8 : Propriété d'approximation de l'équation d'amplitude. Ce chapitre est consacré à la question essentielle de la validité de cette équation d'amplitude. Pour cela, on se restreint au cas cubique, c'est à dire qu'on néglige les termes quadratiques dans la non linéarité de frottement. Nous démontrons dans cette thèse la propriété d'approximation. C'est à dire que l'on souhaite montrer que les ondes modulées générées par les solutions de l'équation de Ginzburg-Landau sont des approximations de solutions de notre problème. Le résultat que nous montrons est le suivant (voir Théorème 8.1) :

Théorème 1.2

Soit une amplitude A solution de l'équation d'amplitude dans une certaine classe de fonction régulière. Alors toute solution de notre problème avec condition initiale déjà proche de l'OPPM modulée générée par A (avec erreur en $O(\varepsilon^{\frac{3}{2}})$), reste proche de l'OPPM modulée (également avec erreur en $O(\varepsilon^{\frac{3}{2}})$), sur une échelle de temps de l'ordre de $O(\varepsilon^2)$ (où l'on a noté ε^2 la distance du paramètre au paramètre critique).

Remarque 1.2. *Le Théorème 8.1 (page 103) est un résultat similaire au Théorème 3.2 dans [GM04] démontré dans le cadre FPU.*

La preuve de ce résultat repose sur les idées suivantes. Nous avons besoin pour commencer de regarder le problème de Cauchy associé à l'équation de Ginzburg-Landau ainsi que la régularité des solutions. Puis nous estimons l'erreur entre les approximations générées par ces amplitudes et les solutions. Enfin, nous appliquons le lemme de Gronwall pour conclure. Pour cette dernière étape, on également besoin d'une estimée du semi-groupe généré par l'opérateur linéaire. Pour cela, on l'écrit explicitement comme transformée de Laplace inverse de la résolvante et on estime à la main la résolvante.

Prolongement de ce travail et problèmes ouverts

Le travail effectué dans cette thèse amène divers prolongements naturels, à la fois d'un point de vue théorique et numérique. Citons les principaux.

- Comme évoqué précédemment, il est possible d'étudier numériquement la stabilité des solutions ondes périodiques du problème I, avec calcul numérique des multiplicateurs de Floquet.

- Dans la partie I, certaines questions restent ouvertes. En particulier, le Théorème de Poincaré-Bendixson nous permet de montrer l'existence d'orbites périodiques dans le cadre du couplage nul. Ce résultat nous donne l'existence mais aucune précision sur la forme des orbites périodiques. Pour prouver le théorème de continuation 2.7 on suppose que cette orbite est de type stick/slip (glissement saccadé) avec motif localisé. Une première étude avec Matlab nous a permis de constater qu'on obtenait effectivement des orbites de ce type dans certaines plages de paramètres. On peut envisager par la suite une étude numérique plus poussée de cette question qui confirmerait ce résultat.

- Dans la partie I, on montre l'existence d'ondes périodiques dans la limite *anti-continue*, c'est à dire qu'on considère un couplage ℓ petit. En d'autres termes, le système tend vers un ensemble de masses isolées. En posant maintenant $\ell = \frac{\zeta}{h}$, où h est la distance à l'équilibre entre deux masses consécutives et $u_j(t) = U(s,t)$ avec $s = jh$, le laplacien discret $\ell^2(u_{j+1} - u_j + u_{j-1})$ tend vers un laplacien continu lorsque h tend vers 0. On obtient dans cette limite *continue* l'équation :

$$\ddot{U} = \zeta^2 \partial_{ss}^2 U - U - F(V + \dot{U}).$$

Une étude numérique de cette équation faite dans [CL89a] montre l'existence de solutions sous forme d'ondes progressives périodiques. Dès lors, on peut s'interroger sur le continuum des ondes progressives trouvées entre ces deux limites.

- Dans le cadre de la loi régularisée dans la partie I, ou bien du système lisse de la partie II, on peut aussi étudier l'existence d'ondes progressives via le Théorème de la Variété Centrale. On obtiendrait un résultat du type : *la dynamique des solutions qui restent petites pour tout temps* $t \in \mathbb{R}$ *de l'équation différentielle avec avance et retard* (2.6) *est décrite par une équation différentielle sur une variété centrale de dimension finie. De plus, une étude de bifurcation pour l'équation sur la variété centrale montre l'existence d'une famille d'orbites périodiques qui bifurquent à partir de* 0. On peut pour cela s'inspirer des obtenus dans [IK98, IK00, JS05]. Notons que cette approche par variété centrale nous permet de trouver des solutions remarquables sous forme de famille d'ondes progressives, mais la question de comment s'articule la dynamique autour de ces ondes progressives est ouverte.

- En ce qui concerne la partie II, outre l'étude du cas avec les termes quadratiques de la non linéarité, nous n'avons pas répondu à toutes les questions sous-jacentes à la théorie de la modulation. Notamment on peut prolonger le résultat d'approximation obtenu par une propriété d'attractivité de l'ensemble de OPPM modulées puis construire une approximation globale à l'aide de pseudo-orbites.

- On peut également envisager d'élargir l'application de ces résultats à d'autres problèmes de tribologie (par exemple dans des cadres comme dans [BC98]).

Première partie

Periodic travelling waves in the Burridge-Knopoff model, combined with a velocity-weakening friction law

Chapitre 2

Introduction

2.1 The Burridge-Knopoff model combined with a velocity-weakening friction law

In geology, a fault is a planar fracture in the rock, across which there is a displacement. Because of the friction and the rigidity, the two rocks in contact cannot simply glide past each other. So stress builds up and when it reaches a level that exceeds a strain threshold, the energy is released and it causes an earthquake. Dry friction can be modelled by the set-valued Coulomb's law. At rest, the friction compensates for the shear stress : the static friction prevents any motion in the system until some critical amount of stress is accumulated. Once this amount is accumulated, the rocks begin to slide.
One of the standard models used to describe an earthquake is the spring-block model, proposed by Burridge and Knopoff [BK67]. It is a one dimensional model, which consists in blocks of same mass, connected to each other by springs of same strength k_c and in contact with a rough fixed surface. The blocks are also connected by springs of strength k_p to an upper surface, moving at constant velocity (figure 2.1). In this model, one of

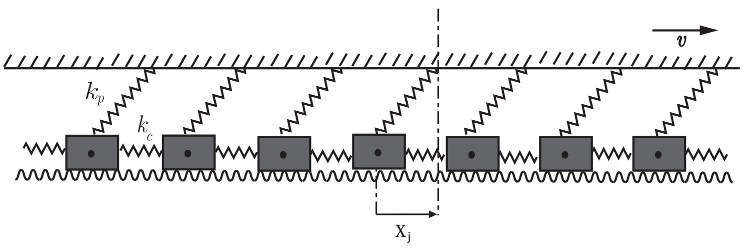


FIGURE 2.1 : Burridge-Knopoff model.

the two rocks is thus discretized (it corresponds to the chain of blocks) and the springs describe the linear response of the contact region to compression and shear.
Several type of friction laws are usually combined with this model. For instance, we can consider friction laws of velocity-weakening type as in [CL89a], for which the friction is

decreasing with increasing speed, or friction laws of type rate and state as in [OK07], for which the friction depends on relative velocity and on a state variable describing the state of the interface. In this part, we will consider the velocity-weakening friction force proposed by Carlson and Langer [CL89a], depending on the relative velocity. The friction ranges between $-F_0$ and F_0 at zero velocity and with the sliding friction decreasing with increasing speed.
Following Carlson and Langer [CL89a], all the physical parameters of the model are spatially uniform. This model is consistent with the Gutenberg-Richter power law, wich says that the average number of earthquakes in any given region and time period decreases exponentially with their magnitude [CL89a, CL89b].
The friction introduces the only nonlinearity in the system and is responsible for the instability that generates chaotic behaviour. The multiple valued character of the friction force at zero velocity causes the system to undergo some stick-slip events : the masses stick and then slip, when the spring force reaches the static friction strength.
This system, combined with several friction laws has been widely numerically studied. The dynamics is very complex. With spatially uniform initial conditions, the system exhibits a uniform periodic stick-slip motion as observed in [CL89b] : all the masses move in unison. This solution appears to be very unstable : if there is a slightly spatial inhomogeneity, the blocks slip approximately at the same time, but the inhomogeneity is amplified during the motion and the system is left in an irregular state, which then gives rise to some smaller events [CL89b]. Most of the events that occur in this model do not involve the whole system, but rather small groups of blocks (localized events). In [SVR93], Schmittbuhl et al. run some computations with periodic boundary conditions, starting with a white noise of small amplitude, and observed travelling waves propagating in the system at constant speed and for which only a small group of blocks are sliding (localized travelling wave). The purpose of this part is to give a proof of the existence of travelling waves in the system.

Denoting by $x_j(t')$ the departure of the block j from its equilibrium position at time t', the equations of motion are

$$m\ddot{x}_j \in k_c(x_{j+1} - 2x_j + x_{j-1}) - k_p x_j - N(v + \dot{x}_j), \quad j \in \mathbb{Z}, \tag{2.1}$$

where v is the constant velocity of the rough surface and N is the nonlinear friction force given by

$$N(y) = N_s\ \Psi(y/v_1),\ \ \Psi(y) = \frac{\mathrm{sgn}(y)}{1 + |y|}, \tag{2.2}$$

where v_1 is a characteristic velocity that determines the scale of the friction law. The friction law is given by figure 2.2. Since it is multivalued at zero velocity, it follows that (2.1) is a *differential inclusion*. We look for travelling waves under the form

$$x_j(t') = x(t),\ \text{where}\ t := j + t'/\tau.$$

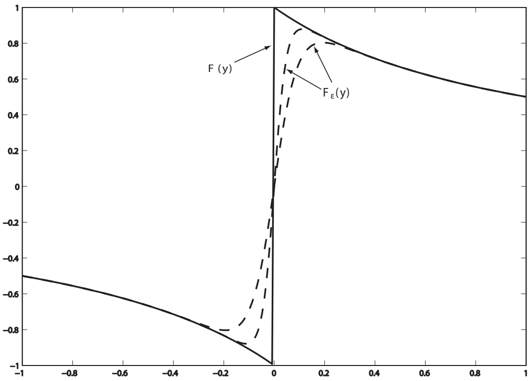


FIGURE 2.2 : Non-smooth friction law F (full line) and smoothened laws F_ε for $\varepsilon = 0.1$ and $\varepsilon = 0.05$ (dashed lines).

Substituting this ansatz into the motion equation leads to

$$\frac{m}{\tau^2}\ddot{x} + k_p x \in k_c(x(t+1) - 2x(t) + x(t-1)) - N(v + \frac{\dot{x}}{\tau}), \tag{2.3}$$

for which we are expecting periodic solutions x travelling at velocity $1/\tau$.

The difficulty is twofold. First we have to deal with a differential *inclusion*. Moreover, since we look for travelling waves, it appears in (2.3) an *advance-and-delay term* : $k_c(x(t+1) - 2x(t) + x(t-1))$.
Aubry and MacKay have introduced the anticontinuous limit concept in halmiltonian discrete lattices (see [Aub97, mKA94]). It has been extended to dissipative systems by Sepulchre and MacKay (see [SmK97]). The technique consists in proving the existence of a solution in a certain class of solutions, by proving its existence in the trivial case $\ell = 0$, and then by proving the persistence of this solution when ℓ is small. Nevertheless, here we have a differential inclusion, so we need to adapt this approach for our problem. We will prove persistence with an adaptated Lyapounov-Schmidt reduction.
A first and natural step to understand this problem is to smoothen the friction law and thus to get rid of the difficulty linked to the differential inclusion. So that we deal with a smooth system, for which we can use the approach of anticontinuous limit and the standard Lyapounov-Schmidt reduction for the persitence. Then we come back to the non-smooth problem and adapt what we have done for the smooth one.
We prove in this first part the existence of periodic travelling waves for both smooth and non-smooth problems in case of weak coupling.

The existence for any value of the coupling parameter is an open problem. We do not raise the problem of the continuum limit, which leads to an PDE. In [Mur99], solutions of the form of shock waves are investigated in the case of a multivalued friction law. Lastly, in [HD98], the continuum limit is explored in case of a time-dependent friction law accounting for ageing.

This first part is organized as follows. Chapter 3 will be devoted to the analysis of the problem for the smoothened equation, and Chapter 4 is devoted to the non-smooth equation. In Section 3.1, we focus on existence of periodic waves for the uncoupled system ($k_c = 0$), and in Section 3.2, we study the case of weakly coupling (k_c close to 0). In Chapter 4, we extend the existence results obtained at Chapter 3 to the nonsmooth system. In Section 4.1 we prove existence of periodic waves in the case of the nonsmooth uncoupled system. And in Section 4.2, we prove persistence of periodic solution in case of weak coupling.

2.2 Statement of the results

We consider equation (2.3) with friction law given by (2.2). To minimize the number of parameters, time and space are rendered dimensionless by defining

$$s = \omega_p t', \quad u_j(s) = \frac{\omega_p}{v_1} x_j(t'),$$

where $\omega_p := \sqrt{\dfrac{k_p}{m}}$ is the pulsation of a single mass without any friction. It then leads to the dimensionless equation

$$\ddot{u}_j \in \ell^2(u_{j+1} - 2u_j + u_{j-1}) - u_j - F(V + \dot{u}_j), \tag{2.4}$$

where $F = F_0\ \Psi$ is the dimensionless friction force with and $F_0 = \dfrac{D_0 \omega_p}{v_1}$ the dimensionless static friction coefficient, $D_0 := \dfrac{N_s}{k_p}$ corresponds to the bigger departure of a single mass in response to the pulling force and V, ℓ are dimensionless parameters with

$$\ell^2 = \frac{k_c}{k_p}, \quad V = \frac{v}{v_1}.$$

The parameter ℓ^2, the ratio of the springs constants, is the *coupling parameter*.

We look for a periodic travelling wave solution of (2.5), and then consider u_j under the form

$$u_j(s) = u(t), \quad \text{where} \quad t = j + s/\tau.$$

Substituting this ansatz in the dimensionless equation (2.4), we obtain

$$\frac{1}{\tau^2}\ddot{u} + u \in \ell^2(u(t+1) - 2u(t) + u(t-1)) - F(V + \frac{\dot{u}}{\tau}). \tag{2.5}$$

The object of this part is then to prove existence of a periodic solution $u(t)$ for this differential inclusion (2.5), which corresponds to a travelling wave propagating at constant velocity $\frac{1}{\tau}$, both in case where the friction law F is smoothened and in the case it is non smoothened. For each case, smooth and nonsmooth, we proceed into two steps :

1. We prove existence of a periodic solution u with no coupling ($\ell = 0$).

2. We prove that we have persistence of this periodic solution in case of weak coupling (ℓ close to zero).

2.2.1 Smoothened problem

In the first part, we consider a smoothened friction law F_ε that approximates the nonsmooth law F. We regularize F the following way

$$F_\varepsilon(y) = \tanh(\frac{y}{\varepsilon}) \cdot |F(y)|\,,\ x \neq 0,\ F_\varepsilon(0) = 0,$$

where ε is a small parameter to be taken in the limit 0 (figure 2.2). The object of Chapter 3 is thus to prove existence of a periodic solution to the second order advance-and-delay differential equation

$$\frac{1}{\tau^2}\ddot{u} + u = \ell^2(u(\xi+1) - 2u(\xi) + u(\xi-1)) - F_\varepsilon(V + \frac{\dot{u}}{\tau}), \tag{2.6}$$

where $\frac{1}{\tau}$, the velocity of the wave, is also considered as an unknown.

Smoothened uncoupled equation

The first step is to prove existence of periodic solution $u(\xi)$ for the uncoupled equation for any value of τ, that is, we set $\ell = 0$ and $\tau = \tau_0 \in \mathbb{R}^+$

$$\frac{\ddot{u}}{\tau_0^2} + u + F_\varepsilon(V + \frac{\dot{u}}{\tau_0}) = 0. \tag{2.7}$$

In Chapter 3, we prove the following theorems, which give existence results of periodic orbits. The first one gives existence in a neighbourhood of the equilibrium point for a value of the parameter V close to a critical value V_c (figure 2.2) and says that the periodic orbit is stable. The second one gives a global result for any value of V.

Theorem 2.1

Let V_c be the unique solution in $\mathbb{R}^+$ of $F_\varepsilon'(y) = 0$ (see figure 2.2). We define $\mu = V - V_c$, the bifurcation parameter. Then we have :

- *If μ is small and $\mu > 0$, the equilibrium is unstable and we have existence of a stable periodic orbit in a neighbourhood of the equilibrium, which radius is of order $\sqrt{\mu}$.*

- *If μ is small and $\mu < 0$, the equilibrium is stable and we do not have any periodic orbit in a neighbourhood of the equilibrium.*

Theorem 2.2

For all $V > V_c$ and for all $\tau_0 \in \mathbb{R}^+$, there exists a periodic orbit for the differential equation (2.7).

In Theorem 2.1, the local existence (μ close to 0) of the periodic orbit comes from a Hopf bifurcation and the global existence of Theorem 2.2, is proved through Poincaré-Bendixson theorem.

Remark 2.3. *Theorem 2.1 gives asymptotic stability for the periodic orbit when $\mu \approx 0$. We will need this stability to prove the existence of a periodic orbit in the coupled case. In the general case $\mu > 0$ the question of stability remains open, but is clear on the numerical computations in the limit $\varepsilon \to 0$ (see figure 3.6).*

Weakly coupled smoothened problem

Let us consider u_0 a periodic orbit for equation (2.7), given by Theorem 2.2, and T_0 its time period. We say that the periodic orbit u_0 is a *non-degenerate orbit*, if 1 is a simple eigenvalue of the monodromy map (or simple Floquet multiplier), and if 1 is isolated in its spectrum (see Definition 1 p.682 in [SmK97]).

The following result gives the existence of a periodic orbit of (2.6) when ℓ is close to 0 as a continuation of a non-degenerate periodic orbit u_0.

Theorem 2.4

Suppose that u_0 is non-degenerate, then for ℓ close to 0, there exists a periodic orbit of (2.5) *close to u_0, with the same time period T_0, and with an inverse velocity τ close to τ_0.*

A more accurate statement of this theorem and its proof will be given in Chapter 3 (Theorem 3.3). The proof is based on the Lyapounov-Schmidt method.

Remark 2.5.

- *We prove this theorem for a more general class of friction laws, including the rate and state friction laws.*
- *We could have fix ℓ and not $T = T_0$, and obtain instead, a family of periodic orbits parametrized by the time period T.*

2.2.2 Nonsmooth problem.

We now focus on the differential inclusion (2.5) and raise the same questions.

Nonsmooth uncoupled problem.

So first, we prove the existence of a periodic orbit for the differential inclusion (2.5) in the particular case $\ell = 0$ and for any $\tau_0 \in \mathbb{R}^+$,

$$\frac{\ddot{u}}{\tau_0^2} + u \in -F(V + \frac{\dot{u}}{\tau_0}). \tag{2.8}$$

Theorem 2.6

For all $V > 0$, there exists a periodic orbit to the differential inclusion (2.8)*.*

The proof is also based on Poincaré-Bendixson theorem which holds in the case of differential inclusion under some non very restrictive hypotheses [Kun00].

Nonsmooth weakly coupled problem

The second part of Chapter 4 is devoted to the nonsmooth weakly coupled problem. We assume that there exists a periodic orbit given by Theorem 2.6, that is of the form stick-slip :

$$\left\{ \begin{array}{lll} \dot{u}(t) & = & -\tau V \text{ in } [0, t_g], \\ \dot{u}(t) & \neq & -\tau V \text{ in }]t_g, T_0[. \end{array} \right. \tag{2.9}$$

This simply means that in a certain time interval $[0, t_g]$, the masses stick to the lower surface, and at $t = t_g$ they begin to move (the strain threshold has been exceeded). Then, we extend the results of Chapter 3 and we prove the following existence result under the following hypotheses $(\boldsymbol{H})$ on the periodic orbit u_0 :

- u_0 is a periodic orbit of type 1 (see Figure 4.3), that is $\dot{u}_0 \geq -\tau_0 V$,

- $T_0 >> 1$,

- $T_0 - t_{g_0} << 1$.

The third hypothesis says that the time during which the masses slide is short compared to the time during which the masses stick. Under the set of these hypotheses, in $]t_g\,, T_0[$, the differential inclusion becomes a differential equation.

Theorem 2.7. *Let u_0 be a T_0-periodic solution of the form* (2.9) *of inclusion* (2.8) *for $\tau = \tau_0 \in \mathbb{R}^+$ and satisfying hypotheses $(\boldsymbol{H})$. Moreover, we assume that we have $u_0(0) = u_0(T_0) < F_0$. We then denote by t_{g0} the time at which the mass begins to slide. Let us also denote by u_{01} and u_{02} the two linearly independent solutions of the linearization of equation* (2.8) *in a neighbourhood of u_0 on the time interval $]t_{g0}, T_0[$, i.e.*

$$\frac{\ddot{u}_{0j}}{\tau_0^2} + u_{0j} + F'\left(V + \frac{\dot{u}_0}{\tau_0}\right)\frac{\dot{u}_{0j}}{\tau_0} = 0, \quad t \in]t_{g0}, T_0[, \quad j \in \{1,2\} \tag{2.10}$$

satisfying the initial conditions $u_{01}(t_{g0}) = 1$, $\dot{u}_{01}(t_{g0}) = 0$ and $u_{02}(t_{g0}) = 0$, $\dot{u}_{02}(t_{g0}) = 1$. Lastly, let us denote by $u_{p,\tilde{h}}$ the solution of the following equation

$$\frac{\ddot{u}}{\tau_0^2} + u + F'\left(V + \frac{\dot{u}_0}{\tau_0}\right)\frac{\dot{u}}{\tau_0} = \tilde{h}, \quad t \in]t_{g0}, T_0[, \tag{2.11}$$

where $\tilde{h} = 2\ddot{u}_0 + \tau_0 F'\left(V + \dfrac{\dot{u}_0}{\tau_0}\right)\dot{u}_0$, satisfying $u_{p,\tilde{h}}(t_{g0}) = \dot{u}_{p,\tilde{h}}(t_{g0}) = 0$.
We then assume that

$$-\tau_0 V \dot{u}_{02}(T_0) + \tau_0 V + \frac{1}{\tau_0^2}\dot{u}_{p,\tilde{h}}(T_0) \neq 0. \tag{CC}$$

Then there exist a neighbourhood $\mathcal{V}$ of 0 in $\mathbb{R}$, and a neighbourhood Ω of (u_0, τ_0, t_{g0}) in $H^2(0, t_{g0}) \times (\mathbb{R}^+)^2$ such that for all $\ell \in \mathcal{V}$, there exists $(u(\ell), \tau(\ell), t_g(\ell))$ in Ω, solution to (2.5) *with $\tau = \tau(\ell)$, with the same period T_0 and of the stick-slip form* (2.9).

To prove this theorem, the strategy previously used for the smoothened weakly coupled system needs to be changed because in the case of differential inclusion we cannot use directly the Implicit Function Theorem (at least the smooth version of it). We are asking periodic orbits of (2.5) to be of the form (2.9), whith $[0, t_g]$ corresponding to the time interval where the masses stick to the lower surface and $]t_g, T_0[$ corresponding to the time interval where the masses slide. So in $]t_g, T_0[$, (2.5) is not an inclusion, but an equation. Moreover we require u to be of class $C^1(\mathbb{R})$ and piecewise $C^2(\mathbb{R})$, which is motivated by the computations.
As in the smoothened case, we also make a Lyapounov-Schmidt reduction to find $u = u_0 + u_1$ on the time interval $]t_g, T_0[$, where t_g is an unknown and u_1 is a small perturbation depending on parameter ℓ. But in this case it is more technical than in the case of Theorem 2.4. Finally, under an hypothesis on the periodic solution u_0, condition (CC), the Lyapounov-Schmidt method gives us $u_1(\ell), \tau(\ell), t_g(\ell)$ with respect to ℓ for $\ell \approx 0$.

Remark 2.8. *To achieve the Lyapounov-Schmidt reduction, we need to satisfy a compatibility condition, which is our hypothesis (CC) in Theorem 2.7. Numerical computations ensure that this condition is generally satisfied (see table 4.1, page 68).*

Chapitre 3

Smoothened Problem

We first study the smoothened problem (2.6). The aim of this chapter is to prove the existence, for some $\tau \in \mathbb{R}^+$, of a periodic function u, solution to equation (2.6) in case of weak coupling ($\ell \approx 0$). For that purpose, we proceed into two steps : first we prove this result without coupling ($\ell = 0$) and then we prove persistence for $\ell \approx 0$.

3.1 Uncoupled smoothened problem ($\ell = 0$)

Here we focus on equation (2.7) and prove that it has a periodic solution u_0 for any value $\tau_0 \in \mathbb{R}^+$ of τ.

3.1.1 Local existence through Hopf bifurcation

We first prove that a Hopf bifurcation gives birth to the periodic orbit. In equation (2.7), we make a scaling in time, given by $s = \tau t$, $x(s) = u(t)$. It leads to equation

$$\ddot{x} + x + F_\varepsilon(V + \dot{x}) = 0, \tag{3.1}$$

which can also be written as

$$\dot{X} = H_\varepsilon(X) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ -F_\varepsilon(V + \dot{x}) \end{pmatrix}, \text{ where } X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}. \tag{3.2}$$

The only equilibrium point is $X_{eq} = \begin{pmatrix} -F_\varepsilon(V) \\ 0 \end{pmatrix}$ and we have $L = DH_\varepsilon(X_{eq}) = \begin{pmatrix} 0 & 1 \\ -1 & -F'_\varepsilon(V) \end{pmatrix}$. For $F'_\varepsilon(V)^2 > 4$, we have two real eigenvalues. We consider in the following that our parameter V, which is positive (since v is positive), is such that $F'_\varepsilon(V)^2 < 2$. This is satisfied if V belongs to an interval $[V_1, +\infty[$ where V_1 is close to zero and tends to zero as ε tends to infinity. We then have two complex conjugated eigenvalues given by $\lambda = \dfrac{-F'_\varepsilon(V) \pm i\sqrt{-\Delta}}{2}$, crossing the imaginary axis when $F'_\varepsilon(V) = 0$.

Let us introduce a parameter $\mu = V - V_c$ where $V_c > V_1$ is the only positive real number y such that $F_\varepsilon'(y) = 0$ (figure 2.2). At $\mu = 0$, we have $F_\varepsilon''(V_c) \neq 0$ and then we have a Hopf bifurcation, since an isolated pair of complex eigenvalues crosses the imaginary axes.

Proof of Theorem 2.1.

Analysis of the bifurcation in [IA98] gives that for $\mu \approx 0$, there exists a periodic orbit in the neighbourhood of the equilibrium point when $a_1\mu > 0$, where a_1 is a coefficient that can be computed with the explicit formula given in [IA98] (pages $154 - 156$). Using this formula we obtain

$$16\, a_1 = -F_\varepsilon^{(3)}(V_c).$$

Thus we check after lengthy but straightforward computations, that for ε small enough, $a_1 < 0$. And in conclusion, there exists a periodic orbit for $\mu \approx 0$ and $\mu > 0$. We also deduce that this periodic orbit is stable and so the Hopf bifurcation is supercritical.

□

We now raise the question of global existence : does this result still holds in the general case $\mu > 0$?

3.1.2 Global existence through Poincaré-Bendixson Theorem

The proof of Theorem 2.2 is based on the Poincaré-Bendixson Theorem [CoLe72], which says that every nonempty compact ω-limit set of a C^1 planar flow, that does not contain an equilibrium point, is a periodic orbit. The idea is to construct a forward invariant domain by the flow. For this, we first prove the following property on the trajectories in the phase space.

Lemma 3.1. *Let $R^2(x, y) = (x - x_0)^2 + y^2$. Then R^2 is decreasing along the trajectories $(x, \dot{x})$ if and only if for all $t \geq 0$ we have*

$$\begin{cases} \dot{x}(t) \leq 0, \\ -F_\varepsilon(\dot{x}(t) + V) \geq x_0, \end{cases} \quad or \quad \begin{cases} \dot{x}(t) \geq 0, \\ -F_\varepsilon(\dot{x}(t) + V) \leq x_0. \end{cases}$$

Proof of the lemma : We have

$$\frac{1}{2}\frac{d}{dt}R^2(x, \dot{x}) = \dot{x}(x - x_0) + \dot{x}\ddot{x} = \dot{x}(-F_\varepsilon(\dot{x} + V) - x_0).$$

□

Proof of Theorem 2.2.
Let us assume that we have a compact forward invariant domain $\mathcal{D}$. We first conclude

the proof of Theorem 2.2.
Given this forward invariant domain $\mathcal{D}$, the ω-limit set of any point in $\mathcal{D}$ is a compact contained in $\mathcal{D}$. Moreover, in our case, there is a unique equilibrium point at $(-F_\varepsilon(V), 0)$, which is repulsive for all $V > V_c$. Thus, except from the equilibrium point, every other ω-limit set consists of regular points and is then a periodic orbit.

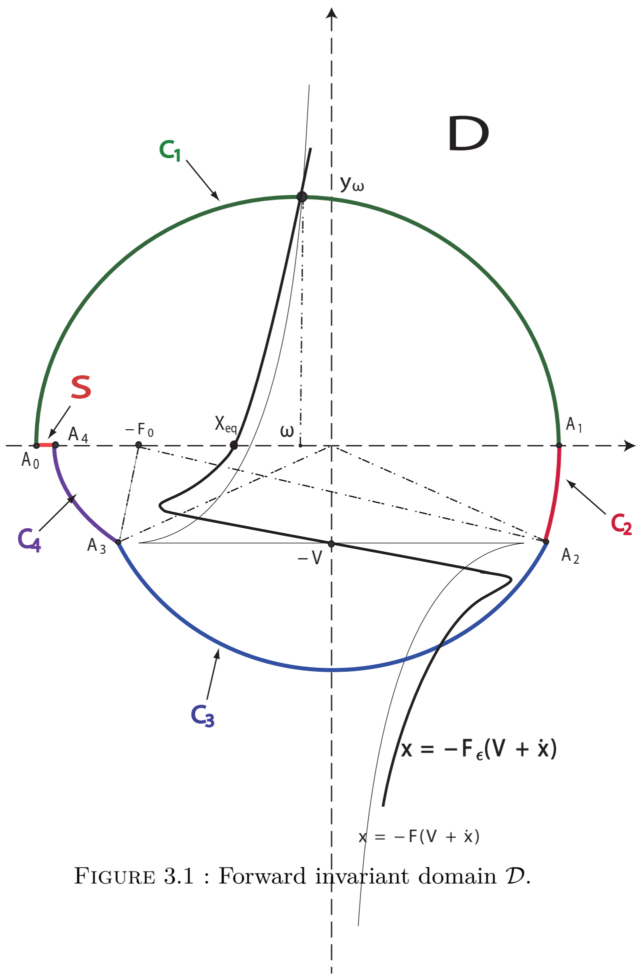


FIGURE 3.1 : Forward invariant domain $\mathcal{D}$.

Construction of $\mathcal{D}$. We define $\mathcal{D}$ by joining arc of circles and segment (see figure 3.1).

• *Step 1.* For $y_\omega \in \,]0, +\infty[$ we define $\omega = -F_\varepsilon(y_\omega + V) = \dfrac{-F_0}{1 + |y_\omega + V|} \tanh(\dfrac{y_\omega + V}{\varepsilon})$.
We notice that, for fixed ε, $w \to 0$ if $y_\omega \to +\infty$.
Let us then consider the arc of circle $\mathcal{C}_1$ of centre $(\omega, 0)$ and radius y_ω joining $A_0(a_0, 0)$ to $A_1(a_1, 0)$, where $a_0 = \omega - y_\omega$ and $a_1 = \omega + y_\omega$. Then for $M(x, \dot{x}) \in \mathcal{C}_1$, we have $\dot{x} \geq 0$ and $-F_\varepsilon(\dot{x} + V) \leq -F_\varepsilon(y_\omega + V) = \omega$ (since $F_\varepsilon(V + .)$ is decreasing in $\mathbb{R}^+$ for $V > V_c$). So using Lemma 3.1 with $x_0 = \omega$, it follows that at each point of $\mathcal{C}_1$, the flow is going into the domain $\mathcal{D}$.

• *Step 2.* Let $\mathcal{C}_2$ be the arc of circle joining A_1 to $A_2(a_2, -V)$, of centre $(-F_0, 0)$ and radius a_1+F_0. Then for all $M(x, \dot{x}) \in \mathcal{C}_2$, we have $\dot{x} \leq 0$ and $-F_\varepsilon(\dot{x}+V) \geq -F_0$. With $x_0 = -F_0$, we deduce that the flow is also going into $\mathcal{D}$. Moreover we have $(a_2 + F_0)^2 + V^2 = (a_1 + F_0)^2$. For y_ω large enough, we have $(a_1 + F_0)^2 - V^2 = (y_\omega + \omega + F_0)^2 - V^2 \geq 0$ and so it follows that $a_2 = -F_0 + \sqrt{(a_1 + F_0)^2 - V^2}$.

• *Step 3.* Let $\mathcal{C}_3$ be the arc of circle of center $(0, 0)$ joining $A_2(a_2, -V)$ to $A_3(a_3, -V)$. Then for all $M(x, \dot{x}) \in \mathcal{C}_3$, we have $\dot{x} \leq 0$ and $-F_\varepsilon(\dot{x} + V) \geq 0$ (since $\dot{x} + V \leq 0$). We again conclude by Lemma 3.1 with $x_0 = 0$. Moreover, we have $a_3 = -a_2 = F_0 - \sqrt{(a_1 + F_0)^2 - V^2} \leq 0$.

• *Step 4.* Let $\mathcal{C}_4$ be the arc of circle of center $(-F_0, 0)$ joining $A_3(a_3, -V)$ to $A_4(a_4, 0)$. We prove that the flow is going into $\mathcal{D}$ as in step 2. Moreover, we have $(a_3+F_0)^2+V^2 = (a_4 + F_0)^2$. Then $a_4 = -F_0 - \sqrt{V^2 + (a_3 + F_0)^2} \leq 0$.

• *Step 5.* Let $\mathcal{S}$ be the segment joining A_4 to A_0. To prove that the domain $\mathcal{D}$ delimited by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{S}$ is forward invariant, it remains to prove that at each point of $\mathcal{S}$, the flow is going into $\mathcal{D}$, *i.e.* $a_0 \leq a_4 \leq 0$

$$\begin{aligned} a_0 - a_4 &= \omega - y_\omega + F_0 + \sqrt{V^2 + (a_3 + F_0)^2}, \\ a_0 - a_4 \leq 0 &\Leftrightarrow \sqrt{V^2 + (a_3 + F_0)^2} \leq y_\omega - \omega - F_0. \end{aligned}$$

Let us take y_ω large enough so that $y_\omega - \omega - F_0 \geq 0$. Then,

$$\begin{aligned} a_0 - a_4 \leq 0 &\Leftrightarrow V^2 + (F_0 + a_3)^2 \leq (y_\omega - \omega - F_0)^2, \\ &\Leftrightarrow V^2 + (2F_0 - \sqrt{(a_1 + F_0)^2 - V^2})^2 \leq (y_\omega - \omega - F_0)^2, \\ &\Leftrightarrow 4F_0^2 - 4F_0\sqrt{(a_1 + F_0)^2 - V^2} + (a_1 + F_0)^2 \leq (y_\omega - \omega - F_0)^2, \\ &\Leftrightarrow 4F_0^2 - 4F_0\sqrt{(a_1 + F_0)^2 - V^2} + (y_\omega + \omega + F_0)^2 \leq (y_\omega - \omega - F_0)^2, \\ &\Leftrightarrow 4F_0^2 - 4F_0\sqrt{(a_1 + F_0)^2 - V^2} + 4y_\omega\omega + 4y_\omega F_0 \leq 0, \\ &\Leftrightarrow F_0^2 + F_0 y_\omega + \omega y_\omega \leq F_0\sqrt{(\omega + y_\omega + F_0)^2 - V^2}. \end{aligned}$$

Moreover it holds

$$F_0^2 + F_0 y_\omega + \omega y_\omega \underset{y_\omega \to +\infty}{\sim} F_0 y_\omega,$$

since we have for fixed ε

$$\omega y_\omega = -\frac{F_0 y_\omega}{1 + |y_\omega + V|}\tanh\left(\frac{y_\omega + V}{\varepsilon}\right) \underset{y_\omega \to +\infty}{\sim} -F_0.$$

It implies that $F_0^2 + F_0 y_\omega + \omega y_\omega \geq 0$ for y_ω large enough. And so we conclude that $a_0 - a_4 \leq 0$ holds if and only if

$$\begin{aligned} & (F_0^2 + F_0 y_\omega + \omega y_\omega)^2 \leq F_0^2((\omega + y_\omega + F_0)^2 - V^2) \\ \Leftrightarrow\ & F_0^4 + 2F_0^2(F_0 + \omega)y_\omega + (F_0 + \omega)^2 y_\omega^2 \leq F_0^4 + 2F_0^3(\omega + y_\omega) + F_0^2(\omega + y_\omega)^2 \\ & \qquad -F_0^2V^2, \\ \Leftrightarrow\ & F_0^2V^2 \leq F_0^2\omega^2 + 2F_0^3\omega - 2F_0\omega y_\omega^2 - \omega^2 y_\omega^2. \end{aligned}$$

Let us define $h(y_\omega) := F_0^2\omega^2 + 2F_0^3\omega - 2F_0\omega y_\omega^2 - \omega^2 y_\omega^2$. We have

$$\lim_{y_\omega \to +\infty} \omega = 0, \; \lim_{y_\omega \to +\infty} y_\omega \omega = -F_0, \; \lim_{y_\omega \to +\infty} \omega^2 y_\omega^2 = F_0^2, \; \lim_{y_\omega \to +\infty} \omega y_\omega^2 = -\infty.$$

Hence we conclude that $\lim_{y_\omega \to \infty} h(y_\omega) = +\infty$, so that there exists y_ω large enough such that $h(y_\omega) \geq F_0^2 V^2$. Thus we can choose $y_\omega \in]0, +\infty[$ so that $a_0 \leq a_4$.

The domain $\mathcal{D}$ is thus a forward invariant domain by the flow for a certain y_ω.

□

Remark 3.2. *The questions of uniqueness and stability of the periodic orbit remain open in the global case despite that we notice it on numerical simulations (see figure 3.2).*

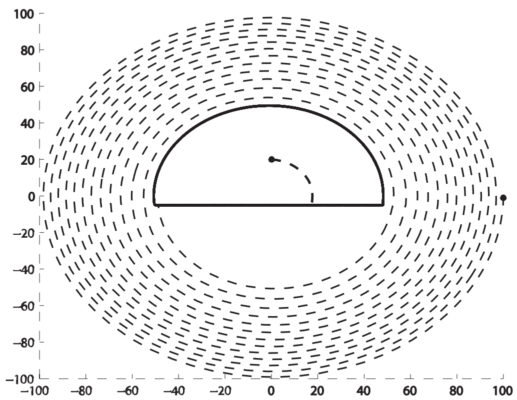


FIGURE 3.2 : Convergence to a same periodic orbit for different initial conditions, for F_ε close to the singular limit F ($V = 5$, $F_0 = 50$, $\varepsilon = 0.001$).

3.1.3 Shape of the periodic solution in the uncoupled smoothened problem

If we plot in the phase space the solutions of equation (2.7) for different values of the parameters V, ε, F_0, we notice that, when ε tends to zero the graph converges to a stick-slip solution (see figure 3.5). The graph of $\dot{x}$ with respect to the time shows also that the slipping is localized when ε tends to zero.

3.2 Weakly coupled smoothened problem ($\ell << 1$)

We come back in this section to the coupled problem for the smoothened friction law (equation (2.6)). We have seen in the previous section that there exists a periodic solution u_0 to equation (2.7) for every $\tau_0 \in \mathbb{R}^+$. Let us denote by T_0 its period. Our aim

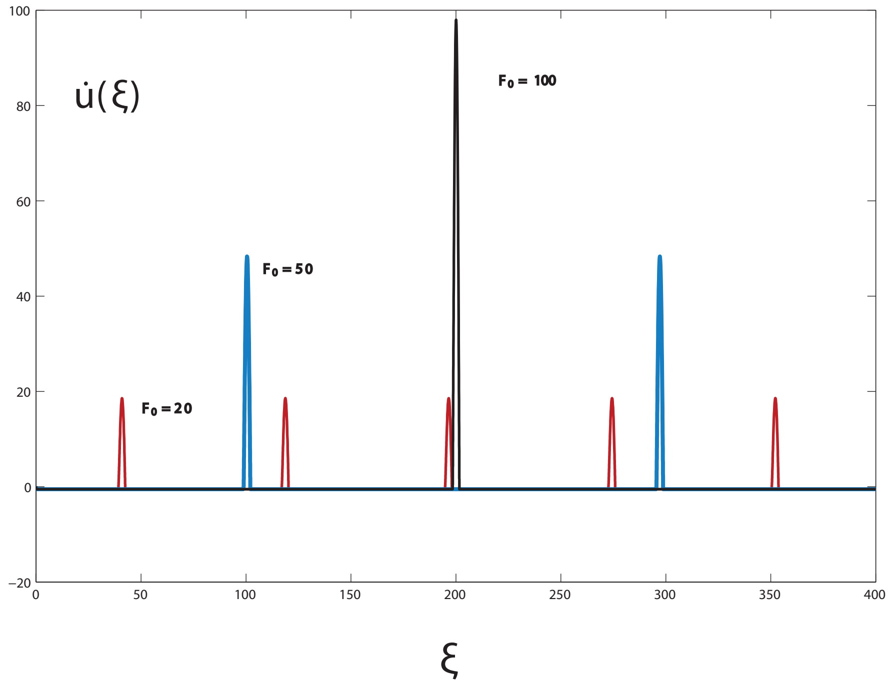


FIGURE 3.3 : Localized travelling waves computed numerically for the smoothened problem, very close to the singular limit $\varepsilon = 0$ ($\varepsilon = 0.001$, $\tau_0 = 1$, $V = 0.5$).

is now to prove that for ℓ close to zero, there exists a periodic solution u to equation (2.6), which is a small perturbation of the periodic solution of (2.7). More precisely, we look for periodic solutions with the same time period T_0 and whose graph is close to the graph of u_0. The existence is given by Theorem 2.4. This section is devoted to the proof of this theorem.

For that purpose, we are going to prove a more general result and consider here instead of our infinite system of equations (2.4), a system where the masses are coupled with a finite number of other masses. Thus we can apply this theorem also for the rate and state friction law of part II. So we consider the following system

$$\dot{U}_n(s) = H(U_n(s)) + \ell^2 \sum_{k\in K} \Psi_k(U_{n-k}), \tag{3.3}$$

where K is a finite set, $U(s) \in \mathbb{R}^d$, H and Ψ_k are smooth functions from $\mathbb{R}^d$ into $\mathbb{R}^d$. This system contains both the rate and state case and the velocity weakening case :

- We recover the velocity weakening friction law with $d = 2$, $U_n = \begin{pmatrix} u_n \\ \dot{u}_n \end{pmatrix}$,

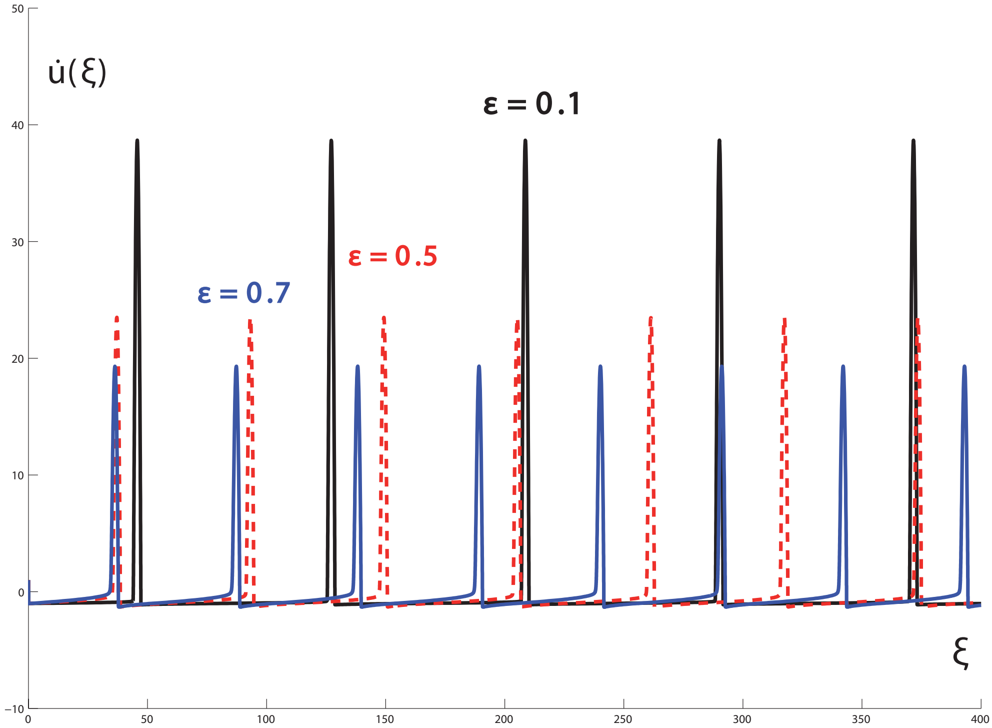


FIGURE 3.4 : Localized travelling waves computed numerically for the smoothened problem, for different values of ε.

$H(U_n) = \begin{pmatrix} \dot{u}_n \\ -u_n - F(\dot{u}_n + V) \end{pmatrix}$ and $\sum_{k \in K} \Psi_k(U_{n-k}) = \begin{pmatrix} 0 \\ u_{n+1} - 2u_n + u_{n-1} \end{pmatrix}$. We then obtain system (2.4).

- With $d = 3$, $U_n = \begin{pmatrix} u_n \\ \dot{u}_n \\ \theta_n \end{pmatrix}$, $H(U_n) = \begin{pmatrix} \dot{u}_n \\ F(U_n) \\ G(U_n) \end{pmatrix}$ and $\sum_{k \in K} \Psi_k(U_{n-k}) = \sum_{k \in K} \begin{pmatrix} 0 \\ \Phi_k(u_{n-k}) \\ 0 \end{pmatrix}$,

we obtain a system under the form

$$\begin{cases} \ddot{u}_n &= \ell^2 \sum_{k \in K} \Phi_k(u_{n-k}) + F(u_n, \dot{u}_n, \theta_n), \\ \dot{\theta}_n &= G(u_n, \dot{u}_n, \theta_n), \end{cases} \tag{3.4}$$

where F is a smooth rate and state friction law depending on the deviation u_n, the velocity $\dot{u}_n$ and on a state variable θ_n satisfying an evolution equation.

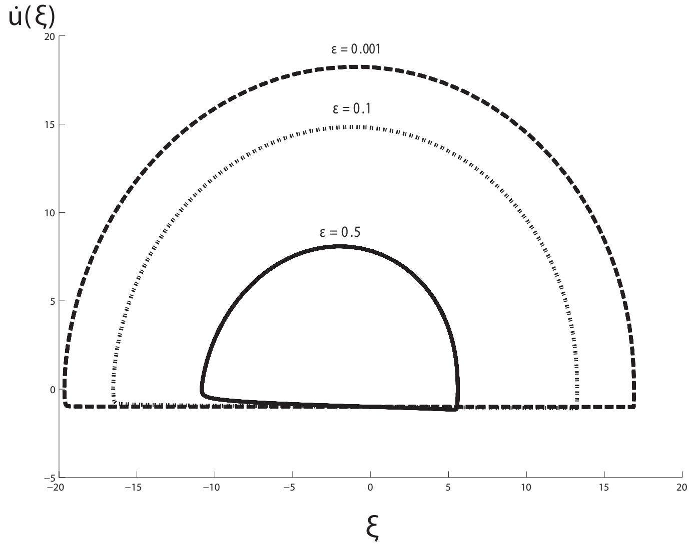


FIGURE 3.5 : Limit cycle of equation (2.7) computed numerically for different values of parameter ε ($F_0 = 20$, $\tau_0 = 1$, $V = 1$).

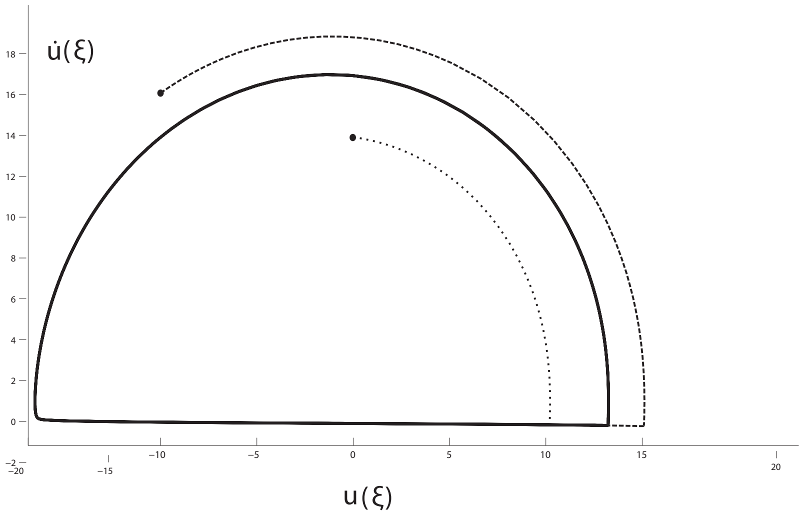


FIGURE 3.6 : Stability of the periodic orbit. Trajectories for different initial conditions ($\varepsilon = 0.1$, $F_0 = 20$, $\tau_0 = 1$, $V = 1$).

As previously, we look for periodic travelling waves, so let us introduce

$$U_n(s) = U(t), \quad t = n + \frac{s}{\tau}.$$

Then U satisfies

$$\frac{1}{\tau}\dot{U}(t) = H(U(t)) + \ell^2 \sum_{k\in K} \Psi_k T_k U(t),$$

where $T_k U(t) = U(t-k)$.

For instance in the rate and state case we have

$$U(t) = \begin{pmatrix} u(t) \\ \frac{1}{\tau}\dot{u}(t) \\ \theta(t) \end{pmatrix},$$

and U satisfies

$$\frac{1}{\tau}\dot{U}(t) = H(U(t)) + \ell^2 \sum_{k\in K} \begin{pmatrix} 0 \\ \Phi_k T_k u(t) \\ 0 \end{pmatrix}. \tag{3.5}$$

Moreover, we make the following hypotheses (H) :

- *There exists a T_0-periodic solution U_0 of* (3.5) *for $\ell = 0$ and for any $\tau = \tau_0 \in \mathbb{R}^+$, that is, U_0 is a solution of*

$$\frac{1}{\tau_0}\dot{U} = H(U). \tag{3.6}$$

- *In addition, assume that* 1 *is a simple Floquet multiplier for the linearized equation*

$$\frac{1}{\tau_0}\dot{U} = DH(U_0)U. \tag{3.7}$$

Let us notice that hypotheses (H) are satisfied in our case : existence has already been proved in Theorem 2.2. For the second condition, it is ensured by the stability of the periodic orbit, which has been proved for small values of the bifurcation parameter μ in Theorem 2.1. In the general case, even if it remains an open question, we have numerically observed the stability (see figure 3.6).
We perform the following calculations with the rate and state system (3.5), but they are not very different in the velocity-weakening case or in the more general case of system (3.3). We now prove the following result, which generalizes Theorem 2.4.

Theorem 3.3

Suppose that hypotheses (H) are satisfied for (U_0, τ_0, T_0). Then there exists a neighbourhood $\mathcal{V}$ of 0 in $\mathbb{R}$ and a neighbourhood Ω of (U_0, τ_0) in $H^2(\mathbb{R}/T_0, \mathbb{R}) \times H^1(\mathbb{R}/T_0, \mathbb{R}) \times H^1(\mathbb{R}/T_0, \mathbb{R}) \times \mathbb{R}$, so that for all $\ell \in \mathcal{V}$, there exists a T_0-periodic solution of (3.5)*, $U(\ell)$, travelling at velocity $\frac{1}{\tau(\ell)}$, in Ω, unique up to phase shift.*

Proof of Theorem 3.3

We look for U with the same period T_0 as U_0 and with a small perturbation of its graph *i.e.* $U = U_0 + U_1$, T_0-periodic. It is simpler to fix the period and to consider the velocity $\dfrac{1}{\tau}$ as a parameter rather than to fix the velocity $\dfrac{1}{\tau_0}$ and consider the period as a parameter, because of the advance-and-delay term. Substituting $U = U_0 + U_1$ in (3.5), we obtain

$$\frac{1}{\tau}(\dot{U}_0 + \dot{U}_1) = H(U_0 + U_1) + \ell^2 \begin{pmatrix} 0 \\ \displaystyle\sum_k \Phi_k T_k(u_0 + u_1) \\ 0 \end{pmatrix},$$

where $U_1 = (u_1, v_1, \theta_1)$. The first equation gives $v_1 = \frac{1}{\tau}\dot{u}_1$. So U_1 is of the form $U_1 = (u_1, \frac{1}{\tau}\dot{u}_1, \theta_1)$.

This equation can be written as

$$g(U_1, \tau, \ell) = 0,$$

where

$$g(U_1, \tau, \ell) = -\frac{1}{\tau}(\dot{U}_1 + \dot{U}_0) + H(U_0 + U_1) + \ell^2 \begin{pmatrix} 0 \\ \displaystyle\sum_k \Phi_k T_k(u_0 + u_1) \\ 0 \end{pmatrix}.$$

So, we see that it would be possible to solve this equation in a neighbourhood of $(0, \tau_0, 0)$ with the Implicit Function Theorem, provided that $Dg(0, \tau_0, 0)$ is invertible, which is not the case. We thus follow the Lyapounov-Schmidt method, which consists in projecting this equation onto some well-chosen spaces.
For that purpose, let us define

$$\begin{cases} \mathscr{L}(U_1) := D_{U_1} g(0, \tau_0, 0).U_1, \\ \mathscr{N}(U_1, \tau, \ell) := \mathscr{L}(U_1) - g(U_1, \tau, \ell). \end{cases}$$

The operator $\mathscr{L}$ is a linear operator from $E := H^2(\mathbb{R}/T_0, \mathbb{R}) \times H^1(\mathbb{R}/T_0, \mathbb{R}) \times H^1(\mathbb{R}/T_0, \mathbb{R})$ into $F := L^2(\mathbb{R}/T_0, \mathbb{R}^3)$. We have

$$\mathscr{L}(U_1) = -\tfrac{1}{\tau_0}\dot{U}_1(\tau_0) + DH(U_0).U_1 =$$

$$\begin{pmatrix} 0 \\ -\frac{1}{\tau_0^2}\ddot{u}_1 + D_1F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).u_1 + \frac{1}{\tau_0}D_2F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\dot{u}_1 + D_3F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\theta_1 \\ -\frac{1}{\tau_0}\dot{\theta}_1 + D_1G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).u_1 + \frac{1}{\tau_0}D_2G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\dot{u}_1 + D_3G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\theta_1 \end{pmatrix},$$

where D_iF denotes the differential with respect to the i-th variable of F.

Equation $g(U_1, \tau, \ell) = 0$ can also be written as

$$\mathscr{L}(U_1) = \mathscr{N}(U_1, \tau, \ell). \tag{3.8}$$

Thus, we have to find the kernel $N(\mathscr{L})$ and the range $R(\mathscr{L})$ of $\mathscr{L}$ in order to project $g(U_1, \tau, \ell) = 0$.

• *Step 1.* **Kernel of $\mathscr{L}$.** We denote by $N(\mathscr{L}_{|E})$ the kernel pf $\mathscr{L}$. The following result holds.

Lemma 3.4. $N(\mathscr{L}_{|E}) = span(\dot{U}_0)$.

Proof : The proof is trivial under the hypotheses (H). Indeed, $N(\mathscr{L})$ corresponds to the T_0-periodic solutions of the linearized equation of the uncoupled problem around U_0 (3.7).

□

• *Step 2.* **Range of $\mathscr{L}$**. We denote by $R(\mathscr{L}_{|E})$ the range of $\mathscr{L}$. Let L be in F. Then, it holds

$$\begin{aligned} L \in R(\mathscr{L}_{|E}) \quad &\Leftrightarrow \quad \exists\, U \ \in E, \quad \frac{1}{\tau_0}\dot{U} - DH(U_0).U = L, \\ &\Leftrightarrow \quad \exists\, U \ \in E, \quad \dot{U} - A(t)U = \tau_0 L(t), \end{aligned}$$

where $A(t) := \tau_0 DH(U_0(t))$ is a T_0-periodic matrix. We shall prove the following.

Lemma 3.5. *The range of $\mathscr{L}_{|E}$, is of codimension* 1 *and is given by* $R(\mathscr{L}_{|E}) = (\mathrm{span}\Phi_0)^{\perp}$, *where Φ_0 is a T_0-periodic solution of an adjoint equation.*

Proof : This result follows directly from the Fredholm's alternative (see [Hal80]for instance), which says

Proposition 3.6 (*Fredholm's alternative*)

Let us consider the following non homogeneous equation :

$$\dot{x} = A(t)x + f(t), \tag{E}$$

where $A(t)$ is the matrix of an endomorphism A on $\mathbb{C}^n$, such that A is continuous and T-periodic, and f is a T-periodic application from $\mathbb{R}$ into $\mathbb{C}^n$. Then there exists a T-periodic solution of equation (E) *if and only if the following compatibility condition is satisfied*

$$\int_0^T \langle y(t), f(t)\rangle_{\mathbb{C}^n}\, dt = 0, \tag{C}$$

for all y such that ${}^t y$ is a T-periodic solution of the adjoint equation

$$\dot{z} = -zA(t). \tag{E_a}$$

Moreover, the space of T-periodic solutions of (E_a) has the same dimension than the space of T-periodic solutions of the homogeneous equation of (E).

This proposition gives existence of periodic solution for a periodic non homogeneous equation provided that some compatibility condition is satisfied. Let us apply this result in dimension 3. The space of T_0-periodic solutions of the homogeneous equation is of dimension 1 (Lemma 3.4). So we deduce that in our case, the space of solutions of the adjoint equation (E_a) is of dimension 1. Let Φ_0 be a T_0-periodic fonction from $\mathbb{R}$ into $\mathbb{R}^3$, such that ${}^t\Phi_0$ is a solution of (E_a). Then Φ_0 spans the space of T_0-periodic solutions to (E_a) :

$$R(\mathscr{L}_{|E}) = \text{span}(\Phi_0)^\perp,$$

where $\perp$ denotes the orthogonal for the usual scalar product in $L^2_{T_0}(\mathbb{R}, \mathbb{R}^3) : \langle f, g\rangle_{L^2_{T_0}(\mathbb{R},\mathbb{R}^3)} = \int_0^T \langle f(t), g(t)\rangle_{\mathbb{R}^3} dt$.

• *Step 3.* **Projection of equation** (3.8)**.**

We recall that we have

$$\begin{aligned} \ker \mathscr{L} &= \text{span}\, \dot{U}_0, \\ R(\mathscr{L}) &= (\text{span}\, \Phi_0)^\perp. \end{aligned}$$

As $\ker \mathscr{L}$ is of finite dimension and $R(\mathscr{L})$ is of finite codimension, we can write

$$\begin{aligned} E &= \ker \mathscr{L} \oplus (\ker \mathscr{L})^\perp = \text{span}\, \dot{U}_0 \oplus (\text{span}\, \dot{U}_0)^\perp, \\ F &= R(\mathscr{L}) \oplus R(\mathscr{L})^\perp = (\text{span}\, \Phi_0)^\perp \oplus \text{span}\, \Phi_0. \end{aligned}$$

Then let us decompose $U_1 = \overline{U}_1 + a\dot{U}_0$, where $a \in \mathbb{R}$ and $\overline{U}_1 \in (\text{span}\, \dot{U}_0)^\perp$. Projecting (3.8) onto span Φ_0 and (span $\Phi_0)^\perp$, we get the two following equations

$$\mathscr{L}(U_1) = \Pi \mathscr{N}(\overline{U}_1 + a\dot{U}_0, \tau, \ell), \tag{3.9}$$

$$0 = (Id - \Pi)\mathscr{N}(\overline{U}_1 + a\dot{U}_0, \tau, \ell), \tag{3.10}$$

where Π is the orthogonal projection onto $R(\mathscr{L})$.

As we have (U_0, τ_0) solution of (3.5), it follows that $g(0, \tau_0, 0) = 0$. Moreover, $\mathscr{L}$ is an isomorphism from span$(\dot{U}_0)^\perp$ into $R(\mathscr{L})$ and $D_{U_1}\mathscr{N}(0, \tau_0, 0).U_1 = 0$ (the linear part of (3.8) is $\mathscr{L}$). So we can solve (3.9) in a neighbourhood of $(0, \tau_0, 0)$ by the Implicit Function Theorem. We thus obtain U_1 with respect to a, τ, ℓ

$$\begin{aligned} \overline{U}_1 &= \overline{U}_1^*(a, \tau, \ell), \\ \overline{U}_1^*(0, \tau_0, 0) &= 0. \end{aligned}$$

And so, in a neighbourhood of $(\overline{U}_1, a, \tau, \ell) = (0, 0, \tau_0, 0)$, we can write

$$U_1 = \overline{U}_1^*(a, \tau, \ell) + a\dot{U}_0.$$

• *Step 4.* **Resolution of the bifurcation equation** (3.10).

The difficulty lies in equation (3.10). We try to compute the parameter τ with respect to ℓ by using again the Implicit Function Theorem. But nothing guarantees that it is possible.
Since $(Id - \Pi)$ is the orthogonal projection onto span Φ_0, equation (3.10) can be written as

$$(Id - \Pi_1)\mathscr{N}(\overline{U}_1^*(a,\tau,\ell) + a\dot{U}_0, \tau, \ell) = 0 \Leftrightarrow \left\langle \mathscr{N}(\overline{U}_1^* + a\dot{U}_0, \tau, \ell), \Phi_0 \right\rangle_{L^2_{T_0}(\mathbb{R},\mathbb{R}^3)} = 0,$$
$$\Leftrightarrow \int_0^{T_0} \left\langle \mathscr{N}(\overline{U}_1^* + a\dot{U}_0, \tau, \ell)(t), \Phi_0(t) \right\rangle_{\mathbb{R}^3} dt = 0.$$

Let us compute $\mathscr{N}$, which is given by

$$\mathscr{N}(U_1, \tau, \ell) \quad = \quad \mathscr{L}(U_1) - g(U_1, \tau, \ell).$$

We recall that

$$g(U_1, \tau, \ell) = -\frac{1}{\tau}(\dot{U}_1 + \dot{U}_0) + H(U_0 + U_1) + \ell^2 \begin{pmatrix} 0 \\ \displaystyle\sum_k \Phi_k T_k(u_0 + u_1) \\ 0 \end{pmatrix}.$$

So $\mathscr{N}$ reads

$$\begin{aligned} \mathscr{N}(U_1, \tau, \ell) \quad = \quad & \frac{1}{\tau}\dot{U}_0 + \begin{pmatrix} \frac{1}{\tau}\dot{u}_1 \\ \frac{1}{\tau^2}\ddot{u}_1 \\ \frac{1}{\tau}\dot{\theta}_1 \end{pmatrix} - \begin{pmatrix} \frac{1}{\tau}(\dot{u}_0 + \dot{u}_1) \\ F(u_0 + u_1, \frac{1}{\tau}(\dot{u}_0 + \dot{u}_1), \theta_0 + \theta_1) \\ G(u_0 + u_1, \frac{1}{\tau}(\dot{u}_0 + \dot{u}_1), \theta_0 + \theta_1) \end{pmatrix} \\ & -\ell^2 \begin{pmatrix} 0 \\ \displaystyle\sum_k \Phi_k T_k(u_0 + u_1) \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ -\frac{1}{\tau_0^2}\ddot{u}_1 + D_1F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).u_1 + \frac{1}{\tau_0}D_2F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\dot{u}_1 \\ \qquad + D_3F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\theta_1 \\ -\frac{1}{\tau_0}\dot{\theta}_1 + D_1G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).u_1 + \frac{1}{\tau_0}D_2G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\dot{u}_1 \\ \qquad + D_3G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\theta_1 \end{pmatrix}. \end{aligned}$$

Let us denote $C(a, \tau, \ell) := \langle \mathscr{N}(\overline{U}_1^* + a\dot{U}_0, \tau, \ell), \Phi_0 \rangle_{L^2_{T_0}(\mathbb{R},\mathbb{R}^3)}$ and $c(\tau) = C(0, \tau, 0)$. We have

$$c(\tau)$$
$$=$$
$$\left\langle \tfrac{1}{\tau}\dot{U}_0 + \begin{pmatrix} -\frac{1}{\tau}\dot{u}_0 \\ (\frac{1}{\tau^2} - \frac{1}{\tau_0^2})\ddot{\overline{u}}_1^*(0,\tau,0) + D_1F.\overline{u}_1^* + \frac{1}{\tau_0}D_2F.\dot{\overline{u}}_1^* \\ \qquad + D_3F.\overline{\theta}_1^* - F(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}^*{}_1), \theta_0 + \overline{\theta}_1^*) \\ (\frac{1}{\tau} - \frac{1}{\tau_0})\dot{\overline{\theta}}_1^*(0,\tau,0) + D_1G.\overline{u}_1^* + \frac{1}{\tau_0}D_2G.\dot{\overline{u}}_1^* \\ \qquad + D_3G.\overline{\theta}_1^* - G(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}_1^*), \theta_0 + \overline{\theta}_1^*) \end{pmatrix}, \Phi_0 \right\rangle,$$

where the differential in F and G are taken at $U_0 = (u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0)$.

Then we have

$$c'(\tau) = \langle -\frac{1}{\tau^2}\dot{U}_0, \Phi_0\rangle + \langle d(\tau), \Phi_0\rangle, \quad \text{where} \quad d(\tau) = \begin{pmatrix} d_1(\tau) \\ d_2(\tau) \\ d_3(\tau) \end{pmatrix},$$

with

$$\begin{aligned}
d_1(\tau) &= \frac{1}{\tau^2}\dot{u}_0, \\
d_2(\tau) &= -\frac{2}{\tau^3}\ddot{\overline{u}}_1^*(0,\tau,0) + (\frac{1}{\tau^2} - \frac{1}{\tau_0^2})\frac{\partial \ddot{\overline{u}}_1^*}{\partial \tau}(0,\tau,0) + D_1F.\frac{\partial \overline{u}_1^*}{\partial \tau}(0,\tau,0) \\
&\quad + \frac{1}{\tau_0}D_2F.\frac{\partial \dot{\overline{u}}_1^*}{\partial \tau}(0,\tau,0) + D_3F.\frac{\partial \overline{\theta}_1^*}{\partial \tau}(0,\tau,0) \\
&\quad - D_1F.(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}_1^*), \theta_0 + \overline{\theta}_1^*).\frac{\partial \overline{u}_1^*}{\partial \tau}(0,\tau,0) \\
&\quad - D_2F(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}_1^*), \theta_0 + \overline{\theta}_1^*).\left\{ -\frac{1}{\tau^2}(\dot{u}_0 + \dot{\overline{u}}_1^*) + \frac{1}{\tau}\frac{\partial \dot{\overline{u}}_1^*}{\partial \tau} \right\} \\
&\quad - D_3F.(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}_1^*), \theta_0 + \overline{\theta}_1^*).\frac{\partial \overline{\theta}_1^*}{\partial \tau}(0,\tau,0),
\end{aligned}$$

$$\begin{aligned}
d_3(\tau) &= -\frac{1}{\tau^2}\dot{\overline{\theta}}_1^*(0,\tau,0) + (\frac{1}{\tau} - \frac{1}{\tau_0})\frac{\partial \dot{\overline{\theta}}_1^*}{\partial \tau}(0,\tau,0) + D_1G.\frac{\partial \overline{u}_1^*}{\partial \tau}(0,\tau,0) \\
&\quad + \frac{1}{\tau_0}D_2G.\frac{\partial \dot{\overline{u}}^*{}_1}{\partial \tau}(0,\tau,0) + D_3G.\frac{\partial \overline{\theta}_1^*}{\partial \tau}(0,\tau,0) \\
&\quad - D_1G.(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}^*{}_1), \theta_0 + \overline{\theta}_1^*).\frac{\partial \overline{u}_1^*}{\partial \tau}(0,\tau,0) \\
&\quad - D_2G(u_0 + \overline{u}_1^*, \frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}^*{}_1), \theta_0 + \overline{\theta}_1^*).\left\{ -\frac{1}{\tau^2}(\dot{u}_0 + \dot{\overline{u}}^*{}_1) + \frac{1}{\tau}\frac{\partial \dot{\overline{u}}_1^*}{\partial \tau} \right\} \\
&\quad - D_3G.(u_0 + \overline{u}_1^*, +\frac{1}{\tau}(\dot{u}_0 + \dot{\overline{u}}_1^*), \theta_0 + \overline{\theta}_1^*).\frac{\partial \overline{\theta}_1^*}{\partial \tau}(0,\tau,0),
\end{aligned}$$

and where as previously, the differential in F and G are taken at U_0. Moreover, for $\tau = \tau_0$, we have $\overline{u}_1^*(0,\tau_0,0) = \dot{\overline{u}}_1^*(0,\tau_0,0) = 0$, so $c'(\tau_0)$ reads

$$\begin{aligned}
c'(\tau_0) &= \left\langle -\frac{1}{\tau_0^2}\dot{U}_0 + \begin{pmatrix} \frac{1}{\tau_0^2}\dot{u}_0 \\ \frac{1}{\tau_0^2}D_2F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\dot{u}_0 \\ \frac{1}{\tau_0^2}D_2G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0).\dot{u}_0 \end{pmatrix}, \Phi_0 \right\rangle, \\
&= \left\langle -\frac{1}{\tau_0}\left[\frac{1}{\tau_0}\dot{U}_0 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & D_2F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0) & 0 \\ 0 & D_2G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0) & 0 \end{pmatrix}.U_0 \right], \Phi_0 \right\rangle.
\end{aligned}$$

This formula enables us to prove

Lemma 3.7. *Under hypotheses (H), we have $c'(\tau_0) \neq 0$.*

Let us admit for the moment the proof of this lemma. To conclude the proof of Theorem 3.3, we notice that equation (3.10) is equivalent to $C(a, \tau, \ell) = 0$. Especially, we have $C(0, \tau_0, 0) = 0$. So using the lemma, we then solve equation (3.10) with the Implicit Function Theorem because $\partial_\tau C(0, \tau_0, 0) = c'(\tau_0) \neq 0$. Thus, in a neighbourhood $\mathcal{V}$ of $(0, \tau_0, 0)$, we have

$$C(a, \tau, \ell) = 0 \Leftrightarrow \tau = \tau^*(a, \ell) \quad \text{with} \tau^*(0, 0) = \tau_0.$$

So we have $U = \overline{U}_1^*(a, \tau^*(a, \ell), \ell) + a\dot{U}_0$.

In conclusion, for all ℓ near 0, there exists a unique one-parameter-family

$$\{(U(t, a, \ell), \tau(a, \ell)), a \in \mathcal{V}(0)\},$$

of T_0-periodic solutions of (3.5), in a neighbourhood of (U_0, τ_0).

$\square$

Remark 3.8. *The parameter a corresponds to the phase shift. Indeed, let us fix ℓ and let us choose $a = 0$, then $(U(t, 0, \ell), \tau(0, \ell))$ is a T_0-periodic solution of (3.5). For all $\phi \in \mathbb{R}$, $(U(t + \phi, 0, \ell), \tau(0, \ell))$ is also a T_0-periodic solution. So by uniqueness, we conclude that there is a bijection between the two following parametered-families*

$$\{(U(t, a, \ell), \tau(a, \ell)), a \in \mathcal{V}(0)\} \quad \text{and} \quad \{(U(t + \phi, 0, \ell), \tau(0, \ell)), \phi \in \mathcal{V}'(0)\}.$$

Proof of Lemma 3.7.

Assume that $c'(\tau_0) = \partial_\tau C(0, \tau_0, 0) = 0$. Then the compatibility condition (C) of the Fredholm's alternative (Proposition 3.6) for the following equation is satisfied

$$\frac{1}{\tau_0}\dot{U} - DH(U_0).U = \frac{1}{\tau_0}\dot{U}_0 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & D_2F(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0) & 0 \\ 0 & D_2G(u_0, \frac{1}{\tau_0}\dot{u}_0, \theta_0) & 0 \end{pmatrix} U_0,$$

where H has been introduced page 35. Let us introduce $X_0(t) = U_0(t\tau_0)$ and $X(t) = U(t\tau_0)$. Then we obtain

$$\dot{X} - DH(X_0).X = \dot{X}_0 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & D_2F(x_0, \dot{x}_0, \psi_0) & 0 \\ 0 & D_2G(x_0, \dot{x}_0, \psi_0) & 0 \end{pmatrix} X_0,$$

wich can also be written as

$$\dot{X} - DH(X_0).X = \dot{X}_0 - DH(X_0).\begin{pmatrix} 0 \\ \dot{x}_0 \\ 0 \end{pmatrix},$$

and this equation admits a T-periodic solution, with $T = \tau_0 T_0$, denoted by $X^* = (x^*, \dot{x}^*, \theta_x^*)$. Let us denote $Z = X^* - \begin{pmatrix} 0 \\ \dot{x}_0 \\ 0 \end{pmatrix}$. Then Z satisfies

$$\begin{aligned}
\dot{Z} &= \begin{pmatrix} \dot{x}^* \\ \ddot{x}^* - \ddot{x}_0 \\ \dot{\theta}_x^* \end{pmatrix}, \\
&= \begin{pmatrix} \dot{x}^* \\ D_1F(X_0).x^* + D_2F(X_0).\dot{x}^* + D_3F(X_0).\theta_x^* + 2\ddot{x}_0 - D_2F(X_0).\dot{x}_0 - \ddot{x}_0 \\ D_1G(X_0).x^* + D_2G(X_0).\dot{x}^* + D_3G(X_0).\theta_x^* + \dot{\theta}_0 - D_2G(X_0).\dot{x}_0 \end{pmatrix}, \\
&= DH(X_0).Z + \dot{X}_0. \qquad (3.11)
\end{aligned}$$

We notice that $t\dot{X}_0$ is solution of this equation, since $\dot{X}_0$ is solution of the linearized equation (3.7). Then solutions of (3.11) are

$$Z(t) = R(t)Z_0 + t\dot{X}_0(t),$$

where $R(t)$ is the fundamental matrix of equation (3.7). Let Z be a T-periodic solution of (3.11). Then, we have

$$Z(T) = Z(0) = Z_0 = R(T)Z_0 + T\dot{X}_0(T). \qquad (3.12)$$

Hence, it follows

$$R(T)Z_0 = Z_0 - T\dot{X}_0(T) = Z_0 - T\dot{X}_0(0).$$

But Z_0 and $\dot{X}_0(0)$ are linearly independent in $\mathbb{R}^3$. Indeed, suppose they are not. So there exists $\mu \in \mathbb{R}$, for which we have $Z_0 = \mu\dot{X}_0(0)$. Then, given (3.12) we deduce

$$Z(T) = Z_0 = \mu\dot{X}_0(0) = \mu R(T)\dot{X}_0(0) + T\dot{X}_0(0). \qquad (3.13)$$

In another hand, $\dot{X}_0$ is a T-periodic solution of the linearized equation (3.7). So, we have

$$R(T).\dot{X}_0(0) = \dot{X}_0(T) = \dot{X}_0(0).$$

And thus we deduce from (3.13) that

$$\mu\dot{X}_0(0) = \mu\dot{X}_0(0) + T\dot{X}_0(0).$$

Hence it follows $T = 0$ and we have a contradiction. Thus we can complete $(\dot{X}_0(0), Z_0)$ in a basis of $\mathbb{R}^3$, in which

$$R(T) = \begin{pmatrix} 1 & -T & * \\ 0 & 1 & * \\ 0 & 0 & * \end{pmatrix}.$$

It implies that 1 is not a simple Floquet multiplier, which contradicts hypotheses (H). It concludes the proof of Lemma 3.7.

□

Chapitre 4

The nonsmooth problem

We now come back to the nonsmooth equation. Let us explain what we mean by 'solution' for the differential inclusion (2.5)

$$\frac{1}{\tau^2}\ddot{u} + u \in \ell^2(u(t+1) - 2u(t) + u(t-1)) - F(V + \frac{\dot{u}}{\tau}).$$

Generally for a differential inclusion $\dot{X} \in F(X)$, the solutions are defined as follows [Smi02] :

Definition 4.1

Consider a differential inclusion of order 1

$$\dot{x}(t) \in F(x(t)), t \in [0, T], \tag{4.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a set-valued map. A solution to (4.1) *is a function x, which is absolutely continuous and which satisfies* (4.1) *almost everywhere in a time interval $I \subset [0, T]$.*

We recall ([Smi02] p.89) that a function $x : [0, T] \to \mathbb{R}^n$ is said absolutely continuous if, given $\varepsilon > 0$, there is $\delta > 0$ such that for any countable collection of disjoint subintervals $[t_k, t'_k]$ of $[0, T]$ satisfying

$$\sum (t'_k - t_k) < \delta,$$

we have

$$\sum |x(t'_k) - x(t_k)| < \varepsilon.$$

An absolutely continuous function is continuous, has bounded variation and is almost everywhere differentiable. Moreover its derivative $\dot{x}$ is a Lebesgue integral function and we have

$$x(t'') - x(t') = \int_{t'}^{t''} \dot{x}(t) dt.$$

In our case, as we are looking for periodic solutions of the form (2.9), we require more regularity on u. So let us precise exactly what we mean by periodic solution of (2.5) :

Definition 4.2

Let u be of class $C^1(\mathbb{R})$ and piecewise $C^2(\mathbb{R})$ and be a T-periodic fonction. Then u is a periodic solution of (2.5) *if u satisfies* (2.5) *almost everywhere in* $[0,T]$.

Such a periodic solution u is a solution in the sense of definition 4.1 if u and $\dot{u}$ are absolutely continuous (since (2.5) is a differential system of order 2). As u is of class C^1, it is lipschitz continuous and it is easy to prove that it is then absolutely continuous using the Mean Value Theorem. The derivative $\dot{u}$ is piecewise C^1 but not C^1. Suppose then that in $[t_k, t'_k]$ there is only a point of discontinuity t_0 for $\ddot{u}$. Then $f_- = \dot{u}_{|[t_k,t_0[}$ is C^1 in $[t_k, t_0[$, and can be extended in $[t_k, t_0]$ by a C^1 function $\tilde{f}_-$. Thus we have

$$f(t_k) - f(t_0) = \tilde{f}_-(t_k) - \tilde{f}_-(t_0) = \int_{t_k}^{t_0} \tilde{f}'_-(t)dt = \int_{[t_k,t_0[} f_-.$$

Similarly, $f_+ = \dot{u}_{|]t_0,t'_k]}$ in $]t_0, t'_k]$ is a C^1 function that can be extended in a C^1 function $\tilde{f}_+$ in $[t_0, t'_k]$. We have

$$f(t_0) - f(t'_k) = \tilde{f}_+(t_0) - \tilde{f}_+(t'_k) = \int_{t_0}^{t'_k} \tilde{f}'_+(t)dt = \int_{]t_0,t'_k]} f'_+.$$

Let us call g_k the function defined by $g_k = f'_-$ in $[t_k, t_0[$, $g_k = f'_+$ in $]t_0, t'_k]$ and $g_k(t_0) = 0$, which is a C^0_m function. Then we have

$$f(t_k) - f(t'_k) = f(t_k) - f(t_0) + f(t_0) - f(t'_k) = \int_{[t_k,t_0[} g_k + \int_{]t_0,t'_k]} g_k = \int_{[t_k,t'_k]} g_k.$$

So we have

$$|f(t_k) - f(t'_k)| \leq \|g_k\|_{\infty,[t_k,t'_k]} |t_k - t'_k| \leq \sup_k \|g_k\|_{\infty,[t_k,t'_k]} |t_k - t'_k|,$$

and thus $f = \dot{u}$ is absolutely continuous.

In the general case, where we have several (but in finite number) discontinuities in $[t_k, t'_k]$, we divide $[t_k, t'_k]$ into subintervals as above.

4.1 Uncoupled problem ($\ell = 0$)

We recall equation (2.8) :

$$\frac{\ddot{u}}{\tau_0} + u \in -F(V + \frac{\dot{u}}{\tau_0}).$$

The existence of a periodic orbit for the uncoupled problem (2.8) is given by Theorem 2.6.

Proof of Theorem 2.6 : in [Dei92], it is proved that the Poincaré-Bendixson Theorem holds for differential planar inclusions if the right-hand side is upper semi continuous (usc), which is our case. So the proof Theorem 2.2 holds for Theorem 2.6.

□

As previously, we make a time scaling and come back to the equation

$$\ddot{x} + x \in -F(V + \dot{x}). \tag{4.2}$$

Let us now look at the shape of a periodic solution x_0 of (4.2) and prove some basic properties, which will be useful in the following to construct a periodic solution of (2.5). We first prove a result of existence of solutions.

Proposition 4.3

For all initial conditions, inclusion (4.2) *has at least one solution x (in the sense of definition 4.1) on $\mathbb{R}^+$.*

Proof : It is given by the existence theorem for differential inclusion with upper semi-continuous (usc) right-hand side (see [Dei92] p.53). We have existence of an absolutely continuous solution on $\mathbb{R}^+$ for a differential inclusion $\dot{X}(t) \in G(X)$, where $G : \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2) \setminus \emptyset$ ($\mathcal{P}(\mathbb{R}^2)$ denotes the powerset of $\mathbb{R}^2$), if the multivalued function G is closed convex, usc, and grows not too fast *i.e.* there exists $c > 0$, such that for all X in $\mathbb{R}^2$ and for all Z in $G(X)$, we have $\|Z\| \leq c(1 + \|X\|)$. We write (4.2) as a first order equation in dimension 2, which reads

$$\dot{X} \in G(X),$$

where $X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ and $G(X) = \begin{pmatrix} \dot{x} \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ -F(V + \dot{x}) \end{pmatrix}$. Obviously, F is closed convex and hence G is also closed convex in $\mathbb{R}^2$. Let us prove that G is usc. It is not really different from the case of the usc function Sgn. Function G is usc if for all A closed in $\mathbb{R}^2$, $G^{-1}(A)$ is closed in $\mathbb{R}^2$. It is equivalent to show that F is usc. So let $[a, b]$ be a segment in $\mathbb{R}$. Then, $F^{-1}([a, b])$ is of the form $\emptyset$, $\{0\} \cup [\alpha, \beta]$, $\{0\} \cup [\alpha, +\infty[$, $\{0\} \cup]-\infty, \beta]$ or $\{0\} \cup]-\infty, \beta] \cup [\alpha, +\infty[$. So in any cases, $F^{-1}([a, b])$ is closed and F usc.
For the growing condition, let us write

$$\|G(X)\|_2 \leq \left\| \begin{pmatrix} \dot{x} \\ -x \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} 0 \\ -F(V + \dot{x}) \end{pmatrix} \right\|_2 \leq \|X\|_2 + \sup_y \left\| \begin{pmatrix} 0 \\ F(y) \end{pmatrix} \right\|_2 \leq \|X\|_2 + M,$$

because F is a bounded multivalued function. So all the hypotheses of the existence theorem in [Dei92] are satisfied and we thus have existence of a solution for any initial conditions.

□

Remark 4.4. *Let us first notice that, as we have a differential inclusion, it would be natural to think that we thus have no uniqueness of the solution. But here we have uniqueness of the solution for the Cauchy Problem : analysis of the phase space (see figure 4.1) shows that we have existence and uniqueness of a solution of* (4.2) *which is C^1 and piecewise C^2. Indeed, as long as x satisfies $\dot{x}(t) = -V$, the right hand side of* (4.2) *is perfectly smooth and so Cauchy-Lipschitz Theorem ensures that the solution exists and is unique. To check that two different solutions cannot pass by a point $(x_0, -V)$, let us* *look at the phase space (figure 4.1). We denote by* $\mathcal{V}(x,\dot{x}) = \begin{pmatrix} \dot{x} \\ -x - F(V+\dot{x}) \end{pmatrix}$ *the vector field. Then we have*

$$\lim_{(x,\dot{x})\to(x_0,-V^+)} \mathcal{V}(x,\dot{x}) = \begin{pmatrix} -V \\ -x_0 - F_0 \end{pmatrix} \quad \text{and} \quad \lim_{(x,\dot{x})\to(x_0,-V^-)} \mathcal{V}(x,\dot{x}) = \begin{pmatrix} -V \\ -x_0 + F_0 \end{pmatrix}.$$

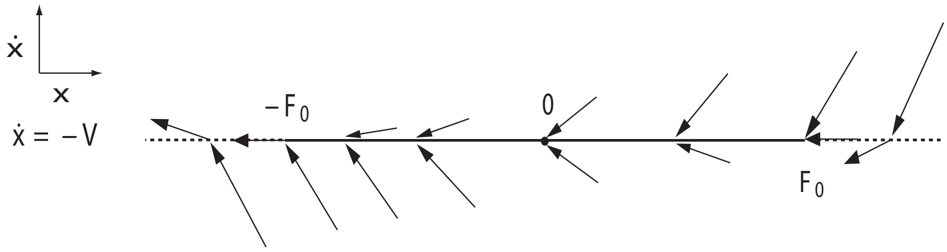


FIGURE 4.1 : Phase space in the neighbourhood of $\dot{x} = -V$.

We see that if x reaches a point $(x_0, -V)$ with $|x_0| > F_0$ then it crosses the line $\dot{x} = -V$. And if it reaches $(x_0, -V)$ with $|x_0| \leq F_0$, the solution x cannot cross the segment $[-F_0, F_0] \times \{-V\}$ since it is attractive : it has to follow it to its end, *i.e.* to the point $(-F_0, -V)$.

Remark 4.5. *We deduce from the analysis of phase space that every solution that reaches the segment* $[-F_0, F_0] \times \{-V\}$ *has to go through its extremity* $(-F_0, -V)$. *This fact is rather interesting. Indeed, as we are looking for periodic solutions of form* (2.9)*, we thus can take* $(-F_0, -V)$ *as initial condition.*

Remark 4.6. *We also notice on the phase space that* x *and* $\dot{x}$ *are continuous. The second derivative* $\ddot{x}$ *is continuous except only at the times when* $\dot{x}$ *reaches* $-V$ *(i.e. at the end of a sliding period). Then* $\ddot{x}$ *is piecewise of class* C^2. *It is what we required in Definition 4.2.*

Let us then prove a result on semi-stability of the periodic orbit (figure 4.2).

Proposition 4.7

Suppose that x_0 *is a* T*-periodic solution of* (4.2) *of the form*

$$\begin{cases} \dot{x}_0(t) &= -V \text{ in } [0, t_{g0}], \\ \dot{x}_0(t) &\neq -V \text{ in }]t_{g0}, T[. \end{cases}$$

Then every other trajectory of (4.2) *with initial values into the domain delimited by* $X_0 = (x_0, \dot{x}_0)$ *reaches the periodic orbit in finite time.*

To prove this proposition, let us first prove the following lemma :

Lemma 4.8. *Let* $M(t) = (x(t), \dot{x}(t))$ *be a trajectory of* (4.2). *Then the distance* $d(M(t), X_{eq})$*, where* X_{eq} *is the unique equilibrium point, is increasing in the half plane* $y > -V$.

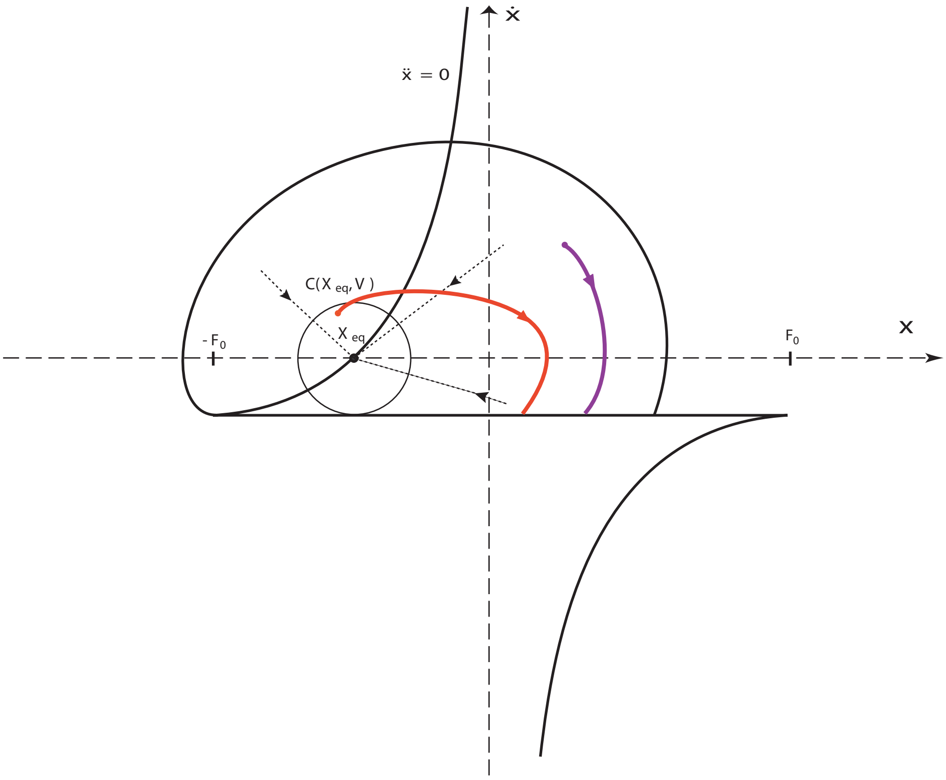


FIGURE 4.2 : Semi-attractivity of the periodic solution

Proof of the lemma : We have $X_{eq} = (-F(V), 0)$. So $d(M(t), X_{eq})^2 = (x(t)+F(V))^2+\dot{x}^2$. And then it follows

$$\frac{d}{dt}\left[d(M, X_{eq})^2\right](t) = 2\dot{x}\left[\ddot{x} + x + F(V)\right].$$

Thus we have

$$\begin{aligned} \frac{d}{dt}\left[d(M, X_{eq})^2\right](t) > 0 \quad &\Leftrightarrow \quad 2\dot{x}\left[-x - F(V+\dot{x}) + x + F(V)\right] > 0, \\ &\Leftrightarrow \quad \dot{x}\left[F(V) - F(V+\dot{x})\right] > 0. \end{aligned}$$

Suppose first that $\dot{x} > 0$. Since we have $F(y) = \dfrac{F_0}{1+y}$ for all $y > 0$, F is decreasing in $\mathbb{R}^{+*}$. So $F(V+\dot{x}) < F(V)$. And then $d(M, X_{eq})$ is increasing.

Assume now that $-V < \dot{x} < 0$, then we have $F(V) < F(V+\dot{x})$ (since $V+\dot{x} > 0$), and consequently, $\dot{x}\left[F(V) - F(V+\dot{x})\right] > 0$. Hence $d(M, X_{eq})$ is also increasing.

□

Proof of Proposition 4.7 : We first consider initial conditions in the domain delimited by X_0 but outside the disk of radius V and centre X_{eq} (see figure 4.2). The disk is tangent to the segment $[-F_0, F_0] \times \{-V\}$, so that it follows from Lemma 4.8 that the trajectory has to cross this segment in finite time.
Let us consider then an initial point in the disk of radius V and centre X_{eq}. Then the distance to X_{eq} is strictly increasing. Suppose that the trajectory does not go out of the disk. It implies that it converges to a limit cycle (there is no attractive equilibrium point into the disk). This limit cycle is necessarily a circle, otherwise the distance of trajectories cannot strictly increase. But no circle can be a solution. Indeed, the distance to X_{eq} is constant in case of a circle trajectory and it contradicts the lemma.
So every trajectory with initial point into the disk has to go out of the disk and so we come back to the previous case.

□

We easily deduce from this proposition the following semi-unicity result.

Proposition 4.9

Suppose $X_0 = (x_0(t), \dot{x}_0(t))$ is a periodic orbit going through the point $(-F_0, -V)$. Then there is no other periodic orbit in the domain delimited by the orbit X_0.

Proof : Suppose that there exists a periodic orbit with initial condition into the domain delimited by the orbit X_0. Then it follows from Proposition 4.7 that it reaches $(x_0(t), \dot{x}_0(t))$ in finite time. So it is not a periodic orbit unless it is X_0.

□

So let us summarize what we already know and what we suppose about the periodic orbit of the uncoupled nonsmooth system :

Theorem 2.6 ensures the existence of a periodic orbit X_0 for the uncoupled nonsmooth system. Uniqueness is not proved but observed on numerical computations in the limit $\varepsilon \to 0$ (see figure 3.6). We assume that in the phase space, this periodic orbit goes through the point $(-F_0, -V)$. In the phase space the shape of this periodic orbit is given by figures 4.3, 4.4 and 4.5. The dashed lines represent the segment $[-F_0, F_0] \times \{V\}$. We consider two different types of periodic orbit as described in figures 4.3 and 4.4.

- Orbits of type 1 (figure 4.3) are periodic trajectories going through the point $(-F_0, -V)$ and for which we have $\dot{x} > -V$. That is, starting at the point $(-F_0, -V)$ the orbit crosses again the line $\dot{x} = -V$ at a point $(y, -V)$ such that $-F_0 < y < F_0$.
- Orbits of type 2 (figures 4.4, 4.5) are periodic trajectories going though the point $(-F_0, -V)$ and for which we have $\dot{x} \leq -V$ for a certain time interval, that is, it crosses the line $\dot{x} = -V$ at a point $(y, -V)$ such that $y > F_0$.

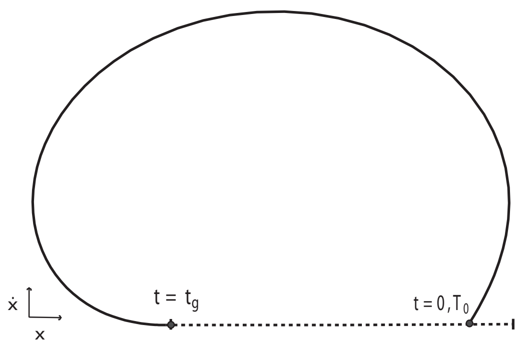


FIGURE 4.3 : Periodic trajectory of type 1

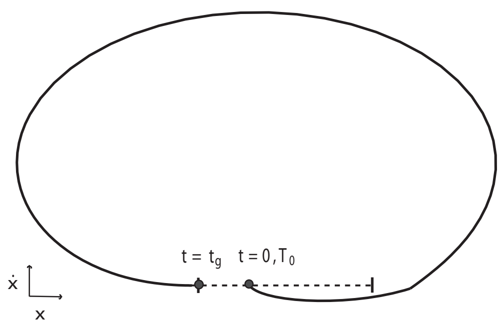


FIGURE 4.4 : Periodic trajectory of type 2

We distinguish the two different cases because in the following we will prove persistence of orbit of type 1 for small ℓ. In numerical computations, the first type is obtained for small values of V (V remains lower than F_0), whereas the second type is obtained for large values of V. Both periodic solutions of type 1 and 2 are C^1 and piecewise C^2. We can even say that these solutions are of class C^2 except at the points kT_0, with $k \in \mathbb{Z}$, for which there is a jump in the second derivative.
In the following, we assume that we have a periodic orbit of type 1. Then we know with Propositions 4.7 and 4.9 that this periodic orbit is semi-attractive and that there is no other periodic orbit inside it.

To conclude with the periodic orbit of the uncoupled problem, let us just notice that the following question remains open : is the solution corresponding to initial condition $(-F_0, -V)$ a periodic orbit ? So that we could possibly have a trajectory of type 3 (figure 4.6) for this initial value condition, which is not periodic.

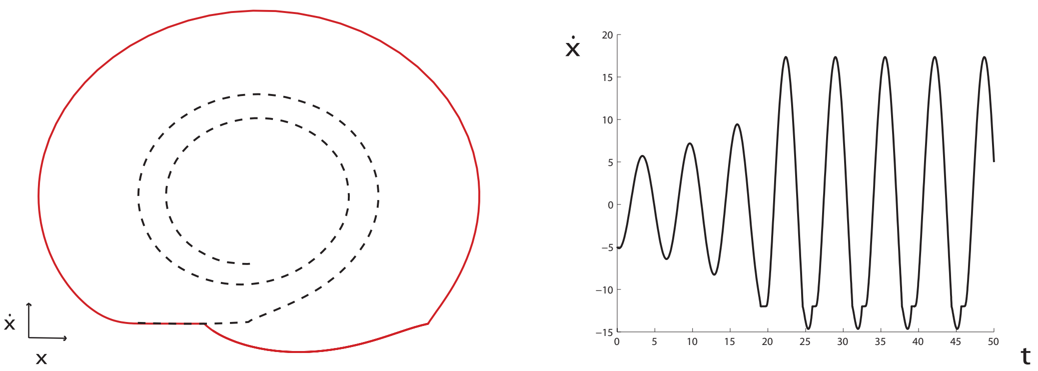


FIGURE 4.5 : Convergence of a solution to a periodic orbit of type 2 (red curve) in the limit $\varepsilon \to 0$ ($F_0 = 10$, $V = 12$, $\varepsilon = 0.001$)

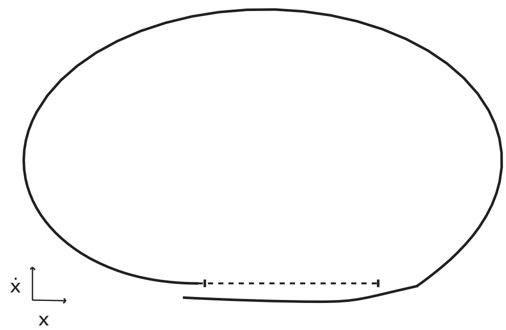


FIGURE 4.6 : Non-periodic trajectory of type 3

4.2 Weakly coupled problem ($\ell << 1$)

This section is devoted to the proof of Theorem 2.7. We recall the differential inclusion (2.5)

$$\frac{\ddot{u}}{\tau^2} + u \in \ell^2\left(u(t+1) - 2u(t) + u(t-1)\right) - F(V + \frac{\dot{u}}{\tau}).$$

Our aim is to prove the existence of a periodic solution u travelling at speed $1/\tau > 0$ for ℓ close to zero. We have previously proved that this equation has a periodic solution for $\ell = 0$. Let u_0 be such a T_0-periodic solution for $\tau = \tau_0$, that is, u_0 is a solution of the uncoupled inclusion (2.8)

$$\frac{\ddot{u}}{\tau_0^2} + u \in -F(V + \frac{\dot{u}}{\tau_0}).$$

We assume that u_0 is of the form

$$\begin{cases} \dot{u}_0(t) &= -\tau_0 V \text{ in } [0, t_{g0}], \\ \dot{u_0}(t) &\neq -\tau_0 V \text{ in }]t_{g0}, T_0[, \end{cases} \tag{4.3}$$

which describes a stick-slip phenomenon : for $t \in [0, t_{g0}]$ the mass sticks and at $t = t_{g0}$, the mass begins to slide. Considering the phase space, there is no reason that a periodic solution should be of that form. We have discussed this point it in Section 4.1. So we assume here that we have an orbit of type 1 and shall prove that for ℓ close to zero, we have a periodic orbit of the similar type.

Particularly, for $t \in]0, t_{g_0}]$, we have $\dot{u}_0(t) = -\tau_0 V$ and $\ddot{u}_0(t) = 0$. So inclusion (2.8) gives that for $t \in]0, t_{g_0}]$, almost everywhere, we have

$$u_0(t) \in [-F_0, F_0].$$

But u_0 being continuous, this implies that

$$\forall t \in [0, t_{g_0}],\ u_0(t) \in [-F_0, F_0].$$

Since $\dot{u}_0(t) < 0$ in $[0, t_{g0}]$, u_0 is decreasing, until we get $u_0(t) = -F_0$ for some t. At this point we also have $\dot{u}_0(t) = -\tau_0 V < 0$, so we see that we cannot satisfy any more the inclusion $u_0(t) \in [-F_0, F_0]$, that means that the mass begins to slip. Thus we have $t = t_{g_0}$ and $u_0(t_{g0}) = -F_0$. We deduce that

$$u_0(t) = -\tau_0 V t + u_0(0)\ , \forall t \in [0, t_{g0}].$$

Moreover, since $u_0(t_{g0}) = -F_0$, we have

$$u_0(0) = -F_0 + \tau_0 V t_{g0}.$$

In the sequel, we need to make the hypotheses :

$$\begin{array}{cc} (H_1) & T_0 >> 1, \\ (H_2) & T_0 - t_{g_0} << 1, \\ (H_3) & u_0 \text{ is an orbit of type 1.} \end{array}$$

That means that the masses slide during a short time interval compared to the time period T_0 of the movement (figures 4.7, 3.3).

Let us now precise our ansatz u. We are looking for a T_0-periodic solution of (2.5) which has the same form than u_0, *i.e.*

$$\left\{ \begin{array}{lcl} \dot{u}(t) & = & -\tau V \text{ in } [0, t_g], \\ \dot{u}(t) & \neq & -\tau V \text{ in }]t_g, T_0[, \end{array} \right. \tag{4.4}$$

where $t_g \approx t_{g0}$ when $\ell \approx 0$ and which is, in addition, close to u_0 for $t \in\ [t_g, T_0]$.

Moreover as for u_0, when the system begins to slip, the sum of the external forces shall be equal to the threshold $-F_0$ *i.e.*

$$\begin{aligned} -F_0 &= u(t_g) - \ell^2 \left(u(t_g-1) - 2u(t_g) + u(t_g+1) \right), \\ &= (1+2\ell^2)u(t_g) - \ell^2(-2\tau V t_g + \tau V T_0 + 2u(0)), \\ &= (1+2\ell^2)(-\tau V t_g + u(0)) - \ell^2(-2\tau V t_g + \tau V T_0 + 2u(0)), \\ &= -\tau V t_g - \ell^2 \tau V T_0 + u(0), \end{aligned}$$

which leads to the ansatz

$$u(0) = -F_0 + \tau V t_g + \ell^2 \tau V T_0.$$

So we have $u(t) = -\tau V t - F_0 + \tau V t_g + \ell^2 \tau V T_0 = \tau V(t_g - t) - F_0 + \ell^2 \tau V T_0$ for $t \in [0, t_g]$.

In conclusion, let us summarize our requirements on u :

- u is a T_0-periodic solution of (2.5), (A_1)
- there exist t_g and τ, such that u is of the form (4.4), (A_2)
- $t_g \approx t_{g_0}$ and $\tau \approx \tau_0$, (A_3)
- $u(t) = \tau V(t_g - t) - F_0 + \ell^2 \tau V T_0, \ \forall t \in [0, t_g]$, (A_4)
- $\|u\|_\infty \leq 2\|u_0\|_\infty$. (A_5)

The unknowns are thus τ, t_g and u in $]t_g, T_0[$. We perform a Lyapounov-Schmidt reduction to determine τ, t_g and u in $]t_g, T_0[$.

Thus, we construct u by part in $[0, T_0]$: u is given by the explicit formula (A_4) in $[0, t_g]$, and we construct u in $]t_g, T_0[$ with the Lyapounov-Schmidt method. We want this solution to be of class C^1 in $\mathbb{R}$ and C^2 in $\mathbb{R}$ except at the points kT_0, $k \in \mathbb{Z}$. For that, we need to satisfy the conditions

$$\begin{aligned} u(t_g^+) &= u(t_g^-), \\ \dot{u}(t_g^+) &= \dot{u}(t_g^-), \\ u(T_0) &= u(0), \\ \dot{u}(T_0) &= \dot{u}(0). \end{aligned}$$

Hence, we have first to check that u given by the formula (A_4) in $[0, t_g]$ is indeed a solution of (2.5) in $[0, t_g]$, which is not obvious considering the advance-and-delay term. This is the aim of the following paragraph. We prove that it is satisfied under the hypotheses (H_1), (H_2) and (H_3), using the fact that u_0 is a solution of (2.8) and that u given by (A_4) is a small perturbation of u_0 in $[0, t_{g0}]$.

4.2.1 Conditions for u to be a solution of (2.5) in $[0, t_g]$.

We have to prove that u given by (A_4) is a solution of equation (2.5).

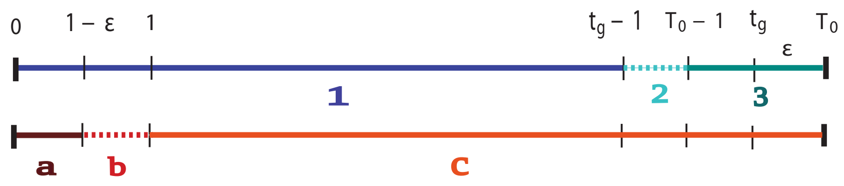


FIGURE 4.7 : Subdivision of $[0, t_g]$.

Conditions for u_0 to be a solution of (2.8) We have assumed that the periodic solution u_0 of (2.8) is of the form (4.3). Especially, we have seen that it implies that $u_0(t) \in [-F_0, F_0]$, $\forall t \in [0, t_{g0}]$. Moreover, it follows from hypothesis (H_3) that there exists $\rho > 0$, such that we more precisely have $u_0(t) \in [-F_0, F_0 - \rho]$, *i.e.* for all t in $[0, t_{g0}]$ we have

$$u_0(t) = \tau_0 V(t_{g0} - t) - F_0 \in [-F_0, F_0 - \rho] \Leftrightarrow 0 \leq \tau_0 V(t_{g0} - t) \leq 2F_0 - \rho. \tag{4.5}$$

This inequality taken at the limit $t \to 0$ gives

$$\tau_0 V t_{g0} \leq 2F_0 - \rho. \tag{C_0}$$

Conditions for u to be a solution of (2.5) **in** $[0, t_g]$ For $t \in]0, t_g]$ we have $\ddot{u} = 0$. So (2.5) gives

$$\forall t \in]0, t_g], \; u(t) - \ell^2(u(t+1) - 2u(t) + u(t-1)) \in -F(0), \tag{4.6}$$

which even holds for all t in $[0, t_g]$, given that u is continuous. We then have to compute $u(t+1) - 2u(t) + u(t-1)$. In $]0, t_g]$, $u(t)$ is given by the explicit formula (A_4), but $t+1$ and $t-1$ may not be in $]0, t_g]$, so that $u(t+1)$ and $u(t-1)$ may be unknown, depending on the value of t. For that reason, we distinguish different cases by dividing the time interval $]0, t_g]$ (see figure 4.7) and prove that u satisfies this equation on each subinterval
.

In figure 4.7, the dashed lines on the first axis represent the times t for which $u(t+1)$ is unknown and the dashed lines on the second axis represent the times t for which $u(t-1)$ is unknown.
For $t \in]0, t_g]$, we have $t + 1 \in]1, t_g + 1]$. We thus have three cases, which reads

- Case 1 : $t + 1 \in]1, t_g]$, i.e. $t \in]0, t_g - 1]$,
- Case 2 : $t + 1 \in [t_g, T_0]$, i.e. $t \in [t_g - 1, T_0 - 1]$,
- Case 3 : $t + 1 \in [T_0, t_g + 1]$, i.e. $t \in [T_0 - 1, t_g]$.

In case 1 and 3, $u(t+1)$ is given by an explicit formula. In case 2, it is unknown. Similarly, $t - 1 \in] - 1, t_g - 1]$. We thus have three cases, which reads

- Case a : $t-1 \in]-1, -\varepsilon]$, i.e. $t \in [0, 1-\varepsilon]$,
- Case b : $t-1 \in [-\varepsilon, 0]$, i.e. $t \in [1-\varepsilon, 1]$,
- Case c : $t-1 \in [0, t_g - 1]$, i.e. $t \in [1, t_g]$,

where we have denoted $\varepsilon = T_0 - t_{g_0}$. In cases a and c, $u(t-1)$ is given by an explicit formula, whereas in case 2, it is unknown. Taking into account the cases 1, 2, 3, a, b, c, we now split $]0, t_g]$ in five subintervals, corresponding to cases 1a, 1b, 1c, 2c, 3c (see figure 4.7).

• Case 1c : $t \in [1, t_g - 1]$. Thus $t,\ t+1,\ t-1 \in\ [0, t_g]$.

Thus here (A_4) holds for $u(t+1)$, $u(t-1)$ and $u(t)$. An easy calculation leads to $u(t+1) - 2u(t) + u(t-1) = 0$. So we have

$$\begin{aligned} (4.6) \quad &\Leftrightarrow \quad u(t) = -\tau V t + u(0) \in [-F_0, F_0],\ \forall t \in [1, t_g - 1], \\ &\Leftrightarrow \quad \tau V (t_g - t) - F_0 + \ell^2 \tau V T_0 \in [-F_0, F_0],\ \forall t \in [1, t_g - 1]. \end{aligned}$$

Hence

$$(4.6) \Leftrightarrow 0 \leq \tau V (t_g - t) + \ell^2 \tau V T_0 \leq 2F_0,\ \forall t \in [1, t_g - 1]. \qquad (C_1)$$

Since $\tau, V, T_0 \geq 0$ and $t_g - t \geq 0$ in $[1, t_g - 1]$, it follows that the lower inequality is satisfied. For the upper inequality it is sufficient to check that it is satisfied at $t = 0$ *i.e.* that we have

$$\tau V t_g + \ell^2 \tau V T_0 \leq 2F_0. \qquad (4.7)$$

Inequality (C_0) gives

$$\tau_0 V t_{g0} \leq 2F_0 - \rho < 2F_0.$$

Since $\tau V t_g$ is close to $\tau_0 V t_{g0}$, for ℓ small enough (4.7) is satisfied and then (C_1) also.

• Case 2c : $t \in [t_g - 1, T_0 - 1[$. Thus we have $t,\ t-1 \in]0, t_g]$, and $t+1 \in\ [t_g, T_0[$.

We deduce from (A_4) that

$$\begin{aligned} u(t) &= -\tau V t + u(0), \\ u(t-1) &= -\tau V (t-1) + u(0), \end{aligned}$$

but we have no explicit expression for $u(t+1)$. Thus,

$$u(t+1) - 2u(t) + u(t-1) = u(t+1) + \tau V t + \tau V - u(0).$$

It follows

$$
\begin{aligned}
&u(t)-\ell^2(u(t+1)-2u(t)+u(t-1)) \in [-F_0,F_0] \Leftrightarrow \\
&\left|-\tau Vt+u(0)-\ell^2\left(u(t+1)+\tau Vt+\tau V-u(0)\right)\right| \le F_0 \Leftrightarrow \\
&0 \le \tau V(t_g-t)+\ell^2\tau VT_0-\ell^2\left(u(t+1)+\tau Vt+\tau V-u(0)\right) \le 2F_0. \qquad (C_2)
\end{aligned}
$$

It follows from (A_5) that

$$|u(t+1)+\tau Vt+\tau V-u(0)| \le 4\|u_0\|_\infty+\tau VT_0+\tau V.$$

To prove (C_2), it is then sufficient to prove the two following inequalities

$$
\begin{gathered}
0 \le \tau V(t_g-t)+\ell^2\tau VT_0-\ell^2 M, \\
\tau V(t_g-t)+\ell^2\tau VT_0+\ell^2 M \le 2F_0,
\end{gathered}
$$

where $M := 4\|u_0\|_\infty+\tau VT_0+\tau V$. As previously, let us check that the first one is satisfied at $t=T_0-1$ and the second one at $t=t_g-1$.

From (A_3) and (H_2) we deduce that $t=T_0-1$, and then it gives $\tau V(t_g-t)+\ell^2\tau VT_0-\ell^2M=\tau V(1-(T_0-t_g)+\ell^2T_0)-\ell^2M \ge 0$. Since $t_g-T_0 << 1$, we have $1-(T_0-t_g)+\ell^2T_0>0$ and then the inequality is satisfied for ℓ close to zero.
With $t=t_g-1$, it gives $\tau V(t_g-t)+\ell^2\tau VT_0+\ell^2M=\tau V+\ell^2\tau VT_0+\ell^2M$. We also deduce from (4.5) that $\tau_0V<\tau_0Vt_{g0}<2F_0$. Hence for ℓ close to zero, and since $|\tau-\tau_0|\to 0$, we have $\tau V+\ell^2\tau VT_0+\ell^2M \le 2F_0$. Thus (C_2) is satisfied for ℓ small enough.

• Case 3c : $t \in [T_0-1,t_g]$. Thus we have t, $t-1 \in]0,t_g]$, and $t+1 \in [T_0,T_0+t_g]$. Thus $u(t)$, $u(t-1)$ and $u(t+1)$ are given by explicit formulas

$$
\begin{aligned}
u(t) &= -\tau Vt+u(0), \\
u(t+1) &= -\tau V(t+1-T_0)+u(0), \\
u(t-1) &= -\tau V(t-1)+u(0),
\end{aligned}
$$

and we deduce

$$
\begin{gathered}
u(t+1)-2u(t)+u(t-1)=\tau VT_0, \\
u(t)-\ell^2(u(t+1)-2u(t)+u(t-1))=\tau V(t_g-t)-F_0.
\end{gathered}
$$

So u is solution of (2.5) if and only if

$$0 \le \tau V(t_g-t) \le 2F_0,\ \forall t \in [T_0-1,t_g]. \qquad (C_3)$$

The lower inequality is clearly satisfied. For the upper inequality, with $t=T_0-1$, we have $\tau V(t_g-t)=\tau V-\tau V(T_0-t_g) \le \tau V$, since $\tau V(T_0-t_g)\ge 0$. Moreover, given (4.5) and given that $t_{g_0} >> 1$ (consequence of (H_1) and (H_2)), we have $\tau_0V << 2F_0$. It holds that $\tau V << 2F_0$ for ℓ close to zero, since $\tau-\tau_0$ is small when ℓ is small. We thus conclude that $\tau V - \tau V(T_0 - t_g) \le \tau V \le 2F_0$ for ℓ close to zero and (C_3) is then satisfied.

• Case 1a : $t \in]0, 1-\varepsilon]$. Then we have t, $t+1 \in]0, t_g]$, and $t-1 \in [-1, -T_0 + t_g] \subset [-T_0, -T_0 + t_g]$. Then it follows

$$\begin{aligned} u(t) &= -\tau V t + u(0), \\ u(t-1) &= -\tau V(t - 1 + T_0) + u(0), \\ u(t+1) &= -\tau V(t+1) + u(0), \end{aligned}$$

and

$$\begin{aligned} u(t+1) - 2u(t) + u(t-1) &= -\tau V T_0, \\ u(t) - \ell^2(u(t+1) - 2u(t) + u(t-1)) &= \tau V(t_g - t) - F_0 + 2\ell^2 \tau V T_0. \end{aligned}$$

So finally we have to prove

$$u(t) + \ell^2 \tau V T_0 \in [-F_0, F_0] \Leftrightarrow 0 \le \tau V(t_g - t) + 2\ell^2 \tau V T_0 \le 2F_0, \ \forall t \in]0, 1-\varepsilon]. \quad (C_4)$$

The lower inequality is clearly satisfied and for the upper inequality, it is the same argument as in the case 3c. At $t = 0$ the inequality is satisfied for ℓ close to zero under hypothesis (C_0).

• Case 1b : $t \in [1-\varepsilon, 1]$ *i.e.* $t - 1 \in [-T_0 + t_g, 0]$. Then t, $t+1 \in]0, t_g]$ and $t - 1 \in [-T_0 + t_g, 0]$ and thus $u(t-1)$ is unknown. We have

$$u(t+1) - 2u(t) + u(t-1) = \tau V t - \tau V - u(0) + u(t-1),$$

and for all t in $[1-\varepsilon, 1]$,

$$\begin{aligned} & u(t) + \ell^2 \tau V T_0 \in [-F_0, F_0] \Leftrightarrow \\ & \qquad 0 \le \tau V(t_g - t) + \ell^2 \tau V T_0 - \ell^2(\tau V(t-1) - u(0) + u(t-1)) \le 2F_0. \end{aligned} \quad (C_5)$$

Once again the lower inequality is clearly satisfied in $[1-\varepsilon, 1]$ for ℓ close to zero and the justification of the upper inequality is similar to the case 2c.

<u>Conclusion</u> : we have proved the following lemma.

Lemma 4.10. *Under hypotheses* $(H_1) - (H_3)$, u *given by* (A_4) *is solution of* (2.5) *in the time interval* $[t_g, T_0]$.

Consequently, we now have to construct u in $[t_g, T_0]$ and to prove that it is a T_0-periodic solution of (2.5) in the sense of definition 4.2.

4.2.2 Condition for u to be a solution of (2.5) in the sliding time interval $[t_g, T_0]$

We make a scaling in time, in order to compare u_0 and u in the same time interval $[t_{g_0}, T_0]$ (u is unknown in $[t_g, T_0]$). Let $s(t) = at + b$ so that $s(t_g) = t_{g0}$ and $s(T_0) = T_0$. We thus have $a = a(t_g) = \dfrac{t_{g0} - T_0}{t_g - T_0}$ and $b = b(t_g) = T_0 \dfrac{t_{g_0} - t_g}{T_0 - t_g}$. Then we define $\overline{u}(s) = u(t)$. The function $\overline{u}$ is defined for $s \in [t_{g0}, T_0]$ and satisfies the differential equation

$$\frac{a^2(t_g)}{\tau^2}\ddot{\overline{u}}(s) + \overline{u}(s) + F(V + \frac{a(t_g)}{\tau}\dot{\overline{u}}) = \ell^2 \left(-2\frac{\tau V s}{a} - 2\overline{u}(s) \right) + C(\tau, t_g, \ell), \ \forall s \in]t_{g0}, T_0[, \quad (4.8)$$

where

$$\begin{aligned} C(t_g, \tau, \ell) &= \ell^2 \left(2\frac{\tau}{a} V b + \tau V T_0 + 2u(0) \right), \\ &= \ell^2 \left(2\frac{\tau}{a} V b + \tau V T_0 - 2F_0 + 2\ell^2 \tau V T_0 + 2\tau V t_g \right). \end{aligned}$$

We are looking for u close to u_0. So let us write

$$\overline{u}(s) = u_1(s) + u_0(s), \ \forall s \in [t_{g0}, T_0].$$

We also require u to be C^1. So we recall that u must satisfy the conditions

$$u(t_g^+) = u(t_g^-), \quad (4.9)$$
$$\dot{u}(t_g^+) = \dot{u}(t_g^-), \quad (4.10)$$
$$u(T_0) = u(0), \quad (4.11)$$
$$\dot{u}(T_0) = \dot{u}(0). \quad (4.12)$$

Equation (4.9) leads to

$$\begin{aligned} \overline{u}(t_{g0}^+) &= \overline{u}(t_{g0}^-) = u(t_g), \\ u_0(t_{g0}) + u_1(t_{g0}^+) &= u_0(t_{g0}) + u_1(t_{g0}^-) = -F_0 + \ell^2 \tau V T_0, \\ u_1(t_{g0}) &= -F_0 + \ell^2 \tau V T_0 + F_0. \end{aligned}$$

So we have

$$\overline{u}_1(t_{g0}) = \ell^2 \tau V T_0.$$

Equation (4.10) leads to

$$\begin{aligned} a\dot{\overline{u}}(t_{g0}^+) &= a\dot{\overline{u}}(t_{g0}^-) = \dot{u}(t_g), \\ \dot{u}_0(t_{g0}) + \dot{u}_1(t_{g0}^+) &= \dot{u}_0(t_{g0}) + \dot{u}_1(t_{g0}^-) = \frac{-\tau V}{a}, \\ \dot{u}_1(t_{g0}) &= \tau_0 V - \frac{\tau V}{a}. \end{aligned}$$

So

$$\dot{u}_1(t_{g0}) = \tau_0 V - \frac{\tau V}{a}.$$

Equation (4.11) leads to

$$\begin{aligned}
\overline{u}(T_0) &= \overline{u}(0) = u(0),\\
u_0(T_0) + u_1(T_0) &= u_0(0) + u_1(0) = -F_0 + \tau V t_g + \ell^2 \tau V T_0,\\
u_1(T_0) &= u_1(0) = u(0) - u_0(0) = -F_0 + \tau V t_g + \ell^2 \tau V T_0 + F_0 - \tau_0 V t_{g0}.
\end{aligned}$$

So

$$u_1(T_0) = V(\tau t_g - \tau_0 t_{g0}) + \ell^2 \tau V T_0.$$

Equation (4.12) leads to

$$\begin{aligned}
a\dot{\overline{u}}(T_0) &= a\dot{\overline{u}}(0) = \dot{u}(0),\\
\dot{u}_0(T_0) + \dot{u}_1(T_0) &= \dot{u}_0(0) + \dot{u}_1(0) = -\frac{\tau V}{a},\\
\dot{u}_1(T_0) &= \tau_0 V - \frac{\tau V}{a}.
\end{aligned}$$

So

$$\dot{u}_1(T_0) = \tau_0 V - \frac{\tau V}{a}.$$

Therefore, let us summarize the four regularity conditions by

$$\begin{aligned}
u_1(t_{g0}) &= \ell^2 \tau V T_0, && (4.13)\\
\dot{u}_1(t_{g0}) &= \tau_0 V - \tfrac{\tau V}{a}, && (4.14)\\
u_1(T_0) &= -\tau_0 V t_{g0} + \tau V t_g + \ell^2 \tau V T_0, && (4.15)\\
\dot{u}_1(T_0) &= \tau_0 V - \tfrac{\tau V}{a}. && (4.16)
\end{aligned}$$

The unknown u_1 must satisfy the following equation

$$\frac{a^2}{\tau^2}(\ddot{u}_1 + \ddot{u}_0) + u_1 + u_0 + F\left(V + \frac{a}{\tau}(\dot{u}_1 + \dot{u}_0)\right) = \ell^2 \left(-2\frac{\tau V s}{a} - 2u_1 - 2u_0\right) + C(\tau, t_g, \ell). \quad (4.17)$$

The Lyapounov-Schmidt method Our aim is now to prove that for all ℓ close to 0, equation (4.17) has a T_0-periodic solution u_1 satisfying conditions (4.13)-(4.16) for some parameters τ, t_g.
For that purpose, let us define $g: \; C^2\left(]t_{g0}, T_0[\right) \times \mathbb{R}^3 \longrightarrow C^0\left(]t_{g0}, T_0[\right) \times \mathbb{R}^4$ by

$$g(u_1, \tau, t_g, \ell) = \begin{pmatrix}
\frac{a^2}{\tau^2}(\ddot{u}_1+\ddot{u}_0)+u_1+u_0+F\left(V+\frac{a}{\tau}(\dot{u}_1+\dot{u}_0)\right)-\ell^2\left(-2\frac{\tau V s}{a}-2u_1-2u_0\right)-C(\tau,t_g,\ell)\\
u_1(t_{g0}^+)-\ell^2\tau V T_0\\
\dot{u}_1(t_{g0}^+)+\frac{\tau V}{a}-\tau_0 V\\
u_1(T_0^-)+\tau_0 V t_{g0}-\tau V t_g-\ell^2\tau V T_0\\
\dot{u}_1(T_0^-)+\frac{\tau V}{a}-\tau_0 V
\end{pmatrix},$$

where $H(s^+)$ and $H(s^-)$ denote the limit of the function H when $t \to s^+$ and $t \to s^-$. Hence we have to solve the problem

$$g(u_1, \tau, t_g, \ell) = 0, \tag{4.18}$$

given that

$$g(0, \tau_0, t_{g0}, 0) = 0. \tag{4.19}$$

Let us then define the linear operator $\mathscr{L}$ which corresponds to the linear part of g, which reads

$$\mathscr{L}(u_1) := D_{u_1} g(0, \tau_0, t_{g0}, 0).u_1 \; .$$

Since $a(t_{g0}) = 1$ we have

$$\mathscr{L}(u_1) = \begin{pmatrix} \frac{\ddot{u}_1}{\tau_0^2} + u_1 + F'(V + \frac{\dot{u}_0}{\tau_0})\frac{\dot{u}_1}{\tau_0} \\ u_1(t_{g0}^+) \\ \dot{u}_1(t_{g0}^+) \\ u_1(T_0^-) \\ \dot{u}_1(T_0^-) \end{pmatrix}, \; u_1 \in C^2(]t_{g0}, T_0[).$$

Kernel of $\mathscr{L}$ Let us prove that we have $\ker\mathscr{L} = \{0\}$.

Suppose that u_1 satisfies $\dfrac{\ddot{u}_1}{\tau_0^2} + u_1 + F'(V + \dfrac{\dot{u}_0}{\tau_0})\dfrac{\dot{u}_1}{\tau_0} = 0$, which is a linear ordinary differential equation on $]t_{g0}, T_0[$ with continuous coefficients ($F'(V + \frac{\dot{u}_0}{\tau_0})$ is in $C^\infty(]t_{g0}, T_0[)$). So for all initial condition $u_1(t_1) = \alpha$, $\dot{u}_1(t_1) = \beta$ with t_1 in $]t_{g0}, T_0[$, this equation admits one and only one solution u_1 defined on $]t_{g0}, T_0[$. If we suppose moreover that $u_1(t_{g0}^+) := \lim\limits_{t \to t_{g0}^+} u_1(t) = 0$ and $u_1(T_0^-) := \lim\limits_{t \to T_0^-} u_1(t) = 0$, then we have $u_1 \equiv 0$. Indeed, F' admits a finite limit when $y \to 0$. So $F'(V + \frac{\dot{u}_0}{\tau_0})$ admits finite limits when $t \to t_{g0}^-$ and $t \to T_0^+$. Let us then consider this equation on the compact $[t_{g0}, T_0]$ where $F'(V + \frac{\dot{u}_0}{\tau_0})$ has been extended. Let us denote $\tilde{u}_1$ the unique solution of the equation for the same initial condition $\tilde{u}_1(t_1) = \alpha$, $\dot{\tilde{u}}_1(t_1) = \beta$. Then we have $\tilde{u}_{1|]t_{g0},T_0[} = u_1$. Moreover, $\tilde{u}_1$ is C^1 on $[t_{g0}, T_0]$ and $u_1, \dot{\tilde{u}}_1$ admit finite limits (all equal to 0) at t_{g0} and T_0. So the limits are equal and we deduce that $\tilde{u}_1(t_{g0}) = \tilde{u}_1(T_0) = 0 = \dot{\tilde{u}}_1(t_{g0}) = \dot{\tilde{u}}_1(T_0)$. Thus, $\tilde{u}_1 \equiv 0$ and consequently $u_1 \equiv 0$.

Range of $\mathscr{L}$ Let us first solve the second order linear equation

$$\frac{\ddot{u}_1}{\tau_0^2} + u_1 + F'\left(V + \frac{\dot{u}_0}{\tau_0}\right)\frac{\dot{u}_1}{\tau_0} = h, \tag{4.20}$$

for a given h in $C^0([t_{g0}, T_0])$. We already know that $\dot{u}_0$ is a solution of the homogeneous equation satisfying the initial values $\dot{u}_0(t_{g0}) = -\tau_0 V = \dot{u}_0(T_0)$ and $\ddot{u}_0(t_{g0}) = 0$. Let

us consider $u_{01} := -\frac{\dot{u}_0}{\tau_0 V}$. Then u_{01} is solution of the homogeneous equation satisfying the initial conditions $u_{01}(t_{g0}) = 1 = u_{01}(T_0)$ and $\dot{u}_{01}(t_{g0}) = 0$. We complete u_{01} into a basis of solutions of the homogeneous equation by choosing u_{02} solution with the initial conditions $u_{02}(t_{g0}) = 0$ and $\dot{u}_{02}(t_{g0}) = 1$.
Given $h \in C^0([t_{g0}, T_0])$, let us now look for a particular solution $u_{p,h}$ of equation (4.20). Let us denote by

$$u_{p,h}(t) = A(t)u_{01}(t) + B(t)u_{02}(t),$$

where $\dot{A}u_{01} + \dot{B}u_{02} = 0$. This function is a solution of (4.20) if and only if

$$\dot{A}\dot{u}_{01} + \dot{B}\dot{u}_{02} = \tau_0^2 h.$$

So (A, B) is solution of

$$\begin{aligned} \dot{A}u_{01} + \dot{B}u_{02} &= 0, \\ \dot{A}\dot{u}_{01} + \dot{B}\dot{u}_{02} &= \tau_0^2 h. \end{aligned}$$

We deduce that

$$\begin{pmatrix} \dot{A} \\ \dot{B} \end{pmatrix} = \frac{1}{w(t)} \begin{pmatrix} -\tau_0^2 h\, u_{02} \\ \tau_0^2 h\, u_{01} \end{pmatrix},$$

where $w(t) = (u_{01}\dot{u}_{02} - \dot{u}_{01}u_{02})(t)$ is the wronskian. It follows

$$u_{p,h}(t) = -\tau_0^2 \left(\int_{t_{g0}}^{t} \frac{u_{02}h}{w} ds \right) u_{01} + \tau_0^2 \left(\int_{t_{g0}}^{t} \frac{u_{01}h}{w} ds \right) u_{02}. \tag{4.21}$$

Let us now come back to the range of the operator $\mathscr{L}$. Then, we have

$$(h, \bar{\alpha}) = (h, (\alpha_1, \alpha_2, \alpha_3, \alpha_4)) \in R(\mathscr{L}),$$

if and only if there exist A and B in $\mathbb{R}$ such that

$$\begin{aligned} u_1(t_{g0}) &= \alpha_1, & (4.22) \\ \dot{u}_1(t_{g0}) &= \alpha_2, & (4.23) \\ u_1(T_0) &= \alpha_3, & (4.24) \\ \dot{u}_1(T_0) &= \alpha_4, & (4.25) \end{aligned}$$

where $u_1 = Au_{01} + Bu_{02} + u_{p,h}$. So we have two degrees of freedom, A, B, and four conditions. So we should have two compatibility conditions. To compute the two conditions, we need the values

$$\begin{aligned} u_{01}(t_{g0}) &= u_{01}(T_0) = 1, \\ \dot{u}_{01}(t_{g0}) &= 0, \\ u_{02}(t_{g0}) &= 0, \\ \dot{u}_{02}(t_{g0}) &= 1, \\ u_{p,h}(t_{g0}) &= \dot{u}_{p,h}(t_{g0}) = 0. \end{aligned}$$

Equation (4.22) gives

$$\alpha_1 = Au_{01}(t_{g0}) + Bu_{02}(t_{g0}) + u_{p,h}(t_{g0}).$$

So we deduce

$$A = \alpha_1.$$

Equation (4.23) gives

$$\alpha_2 = A\dot{u}_{01}(t_{g0}) + B\dot{u}_{02}(t_{g0}) + \dot{u}_{p,h}(t_{g0}).$$

Hence

$$B = \alpha_2.$$

Equation (4.24) gives

$$\alpha_3 = Au_{01}(T_0) + Bu_{02}(T_0) + u_{p,h}(T_0).$$

Thus

$$B = \frac{\alpha_3 - u_{p,h}(T_0) - \alpha_1}{u_{02}(T_0)}.$$

Equation (4.25) gives

$$\alpha_4 = A\dot{u}_{01}(T_0) + B\dot{u}_{02}(T_0) + \dot{u}_{p,h}(T_0).$$

Thus we get

$$B = \frac{\alpha_4 - \dot{u}_{p,h}(T_0) - \alpha_1\dot{u}_{01}(T_0)}{\dot{u}_{02}(T_0)}.$$

So $(h, \bar{\alpha})$ belongs to the range of $\mathscr{L}$ if and only if $\Phi_1(h,\bar{\alpha}) = \Phi_2(h,\bar{\alpha}) = 0$, where

$$\begin{aligned}
\Phi_1(h,\bar{\alpha}) &= \alpha_2 u_{02}(T_0) - \alpha_3 + u_{p,h}(T_0) + \alpha_1, && (4.26)\\
\Phi_2(h,\bar{\alpha}) &= \alpha_2 \dot{u}_{02}(T_0) - \alpha_4 + \dot{u}_{p,h}(T_0) + \alpha_1\dot{u}_{01}(T_0). && (4.27)
\end{aligned}$$

Resolution of equation $g(u_1,\tau,t_g,\ell) = 0$. The equation $g(u_1,\tau,t_g,\ell) = 0$ can be written as $\mathscr{L}(u_1) = \mathscr{N}_g(u_1,\tau,t_g,\ell)$, where $\mathscr{N}_g$ is the nonlinear part of g

$$\mathscr{N}_g(u_1,\tau,t_g,\ell)$$
$$=$$
$$\begin{pmatrix}
(\frac{1}{\tau_0^2} - \frac{a^2}{\tau^2})\ddot{u}_1 - \frac{a^2}{\tau^2}\ddot{u}_0 - u_0 + \ell^2(-2\frac{\tau V s}{a} - 2u_1 - 2u_0) + C(\tau,t_g,\ell) \\
\qquad -F(V + \frac{a}{\tau}(\dot{u}_1 + \dot{u}_0)) + F'(V + \frac{\dot{u}_0}{\tau_0})\frac{\dot{u}_1}{\tau_0} \\
\ell^2 \tau V T_0 \\
V(\tau_0 - \frac{\tau}{a}) \\
V(\tau t_g - \tau_0 t_{g0}) + \ell^2 \tau V T_0 \\
V(\tau_0 - \frac{\tau}{a})
\end{pmatrix}.$$

If we denote by Π the projection onto the range of $\mathscr{L}$, it follows that this equation is equivalent to the system

$$\begin{aligned}\mathscr{L}(u_1) &= \Pi\mathscr{N}_g(u_1,\tau,t_g,\ell), &(4.28)\\ 0 &= (Id-\Pi)\mathscr{N}_g(u_1,\tau,t_g,\ell). &(4.29)\end{aligned}$$

Equation(4.28) can be solved with the Implicit Function Theorem, since $\mathscr{L}$ is bijective from $C^2([t_{g0},T_0])$ into $R(\mathscr{L})$. It follows

$$\begin{aligned}u_1 &= u_1^*(\tau,t_g,\ell),\\ u_1^*(\tau_0,t_{g0},0) &= 0.\end{aligned}$$

The difficulty is to solve equation (4.29) and obtain the parameters τ and t_g as functions of ℓ. This equation is equivalent to the system

$$\Phi_1(\mathscr{N}_g^*(\tau,t_g,\ell)) = 0 = \Phi_2(\mathscr{N}_g^*(\tau,t_g,\ell)), \tag{4.30}$$

where $\mathscr{N}_g^*(\tau,t_g,\ell)=\mathscr{N}_g(u_1^*(\tau,t_g,\ell),\tau,t_g,\ell)$ and Φ_1, Φ_2 are given by (4.26)-(4.27). To solve (4.30), let us introduce the application $\Psi:\mathbb{R}^3\longrightarrow\mathbb{R}^2$ given by

$$\Psi(\tau,t_g,\ell)=\begin{pmatrix}\Phi_1(\mathscr{N}_g^*(\tau,t_g,\ell))\\ \Phi_2(\mathscr{N}_g^*(\tau,t_g,\ell))\end{pmatrix}.$$

Thus we have to solve $\Psi(\tau,t_g,\ell)=0$ in a neighbourhood of $(\tau_0,t_{g0},0)$. To solve it with respect to ℓ by the Implicit Function Theorem, we have to compute $D_{(\tau,t_g)}\Psi(\tau_0,t_{g0},0)$. We denote $\mathscr{N}_g^*(\tau,t_g,\ell)=\begin{pmatrix}h^*\\ \overline{\alpha}\end{pmatrix}$, where

$$\begin{aligned}\alpha &= \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=\begin{pmatrix}\ell^2\tau V T_0\\ V(\tau_0-\frac{\tau}{a})\\ \tau t_g - V(\tau_0 t_{g0})+\ell^2\tau V T_0\\ V(\tau_0-\frac{\tau}{a})\end{pmatrix},\\ h^* &= \left(\frac{1}{\tau_0^2}-\frac{a^2}{\tau^2}\right)\ddot{u}_1^*-\frac{a^2}{\tau^2}\ddot{u}_0-u_0+\ell^2\left(-2\frac{\tau V s}{a}-2u_1^*-2u_0\right)+C(\tau,t_g,\ell)\\ &\quad -F\left(V+\frac{a}{\tau}(\dot{u}_1^*+\dot{u}_0)\right)+F'\left(V+\frac{\dot{u}_0}{\tau_0}\right)\frac{\dot{u}_1^*}{\tau_0}.\end{aligned}$$

It implies

$$\Phi_1(\mathscr{N}_g^*) = V u_{02}(T_0)(\tau_0-\frac{\tau}{a})+V(\tau_0 t_{g0}-\tau t_g)+u_{p,h^*}(T_0), \tag{4.31}$$

$$\Phi_2(\mathscr{N}_g^*) = V\dot{u}_{02}(T_0)(\tau_0-\frac{\tau}{a})+V(\frac{\tau}{a}-\tau_0)+\dot{u}_{p,h^*}(T_0)+\ell^2\tau V T_0\dot{u}_{01}(T_0). \tag{4.32}$$

We recall that $a(t_g)=\dfrac{t_{g0}-T_0}{t_g-T_0}$, so that $a'(t_{g0})=-\dfrac{1}{t_{g0}-T_0}$, $\left(\dfrac{1}{a}\right)'(t_{g0})=\dfrac{1}{t_{g0}-T_0}$, and we notice that at $\ell=0$, we have $\alpha_1=0$, $C(\tau,t_g,\ell)=0$. It implies

$$\begin{aligned}\partial_\tau h^*(\tau_0,t_g,0) &= F'\left(V+\frac{\dot u_0}{\tau_0}\right)\left\{-\frac{1}{\tau_0^2}(\dot u_1^*(\tau_0,t_{g0},0)+\dot u_0)-\frac{1}{\tau_0}\partial_\tau \dot u_1^*(\tau_0,t_{g0},0)\right\}\\ &\quad -\frac{2}{\tau_0^3}(\ddot u_1^*(\tau_0,t_{g0},0)+\ddot u_0)+F'\left(V+\frac{\dot u_0}{\tau_0}\right)\frac{1}{\tau_0}\partial_\tau \dot u_1^*(\tau_0,t_{g0},0).\end{aligned}$$

Since $u_1^*(\tau_0,t_{g0},0)=\dot u_1^*(\tau_0,t_{g0},0)=\ddot u_1^*(\tau_0,t_{g0},0)=0$, it follows

$$\partial_\tau h^*(\tau_0,t_g,0)=-\frac{2}{\tau_0^3}\ddot u_0+\frac{1}{\tau_0^2}F'\left(V+\frac{\dot u_0}{\tau_0}\right)\dot u_0.$$

Similarly, we have

$$\begin{aligned}\partial_{t_g} h^*(\tau_0,t_{g0},0) &= F'\left(V+\frac{\dot u_1^*(\tau_0,t_{g0},0)+\dot u_0}{\tau_0}\right)\left\{\frac{\dot u_1^*(\tau_0,t_{g0},0)+\dot u_0}{\tau_0(t_{g0}-T_0)}-\frac{1}{\tau_0}\partial_{t_g}\dot u_1^*(\tau_0,t_{g0},0)\right\}\\ &\quad +F'\left(V+\frac{\dot u_0}{\tau_0}\right)\frac{1}{\tau_0}\partial_{t_g}\dot u_1^*(\tau_0,t_{g0},0)+\frac{1}{\tau_0^2}\frac{2}{t_{g0}-T_0}(\ddot u_1^*(\tau_0,t_{g0},0)+\ddot u_0),\\ &= \frac{2}{\tau_0^2(t_{g0}-T_0)}\ddot u_0+F'\left(V+\frac{\dot u_0}{\tau_0}\right)\frac{\dot u_0}{\tau_0(t_{g0}-T_0)}.\end{aligned}$$

We recall that $u_{p,h}$ is given by (4.21)

$$u_{p,h}(T_0)=-\tau_0^2u_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{u_{02}h}{w}ds+\tau_0^2u_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{u_{01}h}{w}ds.$$

So we have

$$\begin{aligned}\partial_\tau u_{p,h^*}(T_0)(\tau_0,t_{g0},0) &= -\tau_0^2u_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{u_{02}}{w}\partial_\tau h^*+\tau_0^2u_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{u_{01}}{w}\partial_\tau h^*ds\\ &= -\tau_0^2u_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{u_{02}}{w}\left\{-\frac{2}{\tau_0^3}\ddot u_0-\frac{1}{\tau_0^2}F'\left(V+\frac{\dot u_0}{\tau_0}\right)\dot u_0\right\}ds\\ &\quad +\tau_0^2u_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{u_{01}}{w}\left\{-\frac{2}{\tau_0^3}\ddot u_0-\frac{1}{\tau_0^2}F'\left(V+\frac{\dot u_0}{\tau_0}\right)\dot u_0\right\}ds\\ &= \frac{u_{01}(T_0)}{\tau_0}\int_{t_{g0}}^{T_0}\frac{u_{02}}{w}\left\{2\ddot u_0+\tau_0F'\left(V+\frac{\dot u_0}{\tau_0}\right)\dot u_0\right\}ds\\ &\quad -\frac{u_{02}(T_0)}{\tau_0}\int_{t_{g0}}^{T_0}\frac{u_{01}}{w}\left\{2\ddot u_0+\tau_0F'\left(V+\frac{\dot u_0}{\tau_0}\right)\dot u_0\right\}ds.\end{aligned}$$

Let us denote by

$$I_1 = \int_{t_{g0}}^{T_0}\frac{u_{01}}{w}\left\{2\ddot u_0+\tau_0F'(V+\frac{\dot u_0}{\tau_0})\dot u_0\right\}ds, \tag{4.33}$$

$$I_2 = \int_{t_{g0}}^{T_0}\frac{u_{02}}{w}\left\{2\ddot u_0+\tau_0F'(V+\frac{\dot u_0}{\tau_0})\dot u_0\right\}ds, \tag{4.34}$$

so that we have

$$\partial_\tau u_{p,h^*}(T_0)(\tau_0,t_{g0},0)=\frac{u_{01}(T_0)}{\tau_0}I_2-\frac{u_{02}(T_0)}{\tau_0}I_1.$$

Similarly,

$$\begin{aligned}
\partial_{t_g} u_{p,h^*}(T_0)(\tau_0, t_{g0}, 0) &= -\tau_0^2 u_{01}(T_0) \int_{t_{g0}}^{T_0} \frac{u_{02}}{w} \partial_{t_g} h^* ds + \tau_0^2 u_{02}(T_0) \int_{t_{g0}}^{T_0} \frac{u_{01}}{w} \partial_{t_g} h^* ds \\
&= -\tau_0^2 u_{01}(T_0) \int_{t_{g0}}^{T_0} \frac{u_{02}}{w} \left\{ \frac{2\ddot{u}_0}{\tau_0^2 (t_{g0} - T_0)} + \right. \\
&\qquad \left. F'\left(V + \frac{\dot{u}_0}{\tau_0}\right) \frac{\dot{u}_0}{\tau_0 (t_{g0} - T_0)} \right\} ds \\
&\quad + \tau_0^2 u_{02}(T_0) \int_{t_{g0}}^{T_0} \frac{u_{01}}{w} \left\{ -\frac{2}{\tau_0^3} \ddot{u}_0 - \frac{1}{\tau_0^2} F'\left(V + \frac{\dot{u}_0}{\tau_0}\right) \dot{u}_0 \right\} ds \\
&= -\frac{u_{01}(T_0)}{t_{g0} - T_0} \int_{t_{g0}}^{T_0} \frac{u_{02}}{w} \left\{ 2\ddot{u}_0 + \tau_0 F'\left(V + \frac{\dot{u}_0}{\tau_0}\right) \dot{u}_0 \right\} ds \\
&\quad + \frac{u_{02}(T_0)}{t_{g0} - T_0} \int_{t_{g0}}^{T_0} \frac{u_{01}}{w} \left\{ 2\ddot{u}_0 + \tau_0 F'\left(V + \frac{\dot{u}_0}{\tau_0}\right) \dot{u}_0 \right\} ds.
\end{aligned}$$

So we have

$$\partial_{t_g} u_{p,h^*}(T_0)(\tau_0, t_{g0}, 0) = -\frac{u_{01}(T_0)}{t_{g0} - T_0} I_2 + \frac{u_{02}(T_0)}{t_{g0} - T_0} I_1.$$

We recall that

$$\dot{u}_{p,h} = -\tau_0^2 \dot{u}_{01} \int_{t_{g0}}^{t} \frac{u_{02} h}{w} ds + \tau_0^2 \dot{u}_{02} \int_{t_{g0}}^{t} \frac{u_{01} h}{w} ds.$$

So, we obtain in the same way

$$\partial_\tau \dot{u}_{p,h^*}(T_0)(\tau_0, y_{g0}, 0) = \frac{\dot{u}_{01}(T_0)}{\tau_0} I_2 - \frac{\dot{u}_{02}(T_0)}{\tau_0} I_1,$$

$$\partial_{t_g} \dot{u}_{p,h^*}(T_0)(\tau_0, y_{g0}, 0) = -\frac{\dot{u}_{01}(T_0)}{t_{g0} - T_0} I_2 + \frac{\dot{u}_{02}(T_0)}{t_{g0} - T_0} I_1.$$

We now come back to the computation of the partial derivative of Φ_1 and Φ_2. Given (4.31)-(4.32) and the above computations, we have

$$\begin{aligned}
\partial_\tau \Phi_1(\tau_0, t_{g0}, 0) &= V u_{02}(T_0) + V t_{g0} + \partial_\tau u_{p,h^*}(T_0)(\tau_0, t_{g0}, 0), \\
&= V u_{02}(T_0) + V t_{g0} + \frac{1}{\tau_0} u_{01}(T_0) I_2 - \frac{1}{\tau_0} u_{02}(T_0) I_1,
\end{aligned}$$

$$\begin{aligned}
\partial_\tau \Phi_2(\tau_0, t_{g0}, 0) &= V \dot{u}_{02}(T_0) - V + \partial_\tau \dot{u}_{p,h^*}(T_0)(\tau_0, t_{g0}, 0), \\
&= V \dot{u}_{02}(T_0) - V + \frac{1}{\tau_0} \dot{u}_{01}(T_0) I_2 - \frac{1}{\tau_0} \dot{u}_{02}(T_0) I_1,
\end{aligned}$$

$$\begin{aligned}
\partial_{t_g} \Phi_1(\tau_0, t_{g0}, 0) &= -\frac{\tau_0 V}{t_{g0} - T_0} u_{02}(T_0) - V\tau_0 + \partial_{t_g} u_{p,h^*}, \\
&= -\frac{\tau_0 V}{t_{g0} - T_0} u_{02}(T_0) - V\tau_0 - \frac{1}{t_{g0} - T_0} u_{01}(T_0) I_2 + \frac{1}{t_{g0} - T_0} u_{02}(T_0) I_1,
\end{aligned}$$

$$
\begin{aligned}
\partial_{t_g}\Phi_2(\tau_0, t_{g0}, 0) &= -\frac{\tau_0 V}{t_{g0}-T_0}\dot{u}_{02}(T_0) + \frac{V\tau_0}{t_{g0}-T_0} + \partial_{t_g}\dot{u}_{p,h^*},\\
&= -\frac{\tau_0 V}{t_{g0}-T_0}\dot{u}_{02}(T_0) + \frac{V\tau_0}{t_{g0}-T_0} - \frac{\dot{u}_{01}(T_0)}{t_{g0}-T_0}I_2 + \frac{\dot{u}_{02}(T_0)}{t_{g0}-T_0}I_1.
\end{aligned}
$$

We can now proceed to the computation of the determinant of $D_{(\tau,t_g)}\Psi(\tau_0, t_{g0}, 0)$

$$
\det D_{(\tau,t_g)}\Psi(\tau_0, t_{g0}, 0) = \left(\partial_\tau\Phi_1.\partial_{t_g}\Phi_2 - \partial_\tau\Phi_2.\partial_{t_g}\Phi_1\right)(\tau_0, t_{g0}, 0),
$$

$$
\begin{aligned}
\det D_{(\tau,t_g)}\Psi(\tau_0, t_{g0}, 0) = {} & \Big(V u_{02}(T_0) + V t_{g0} + \tfrac{u_{01}(T_0)}{\tau_0}I_2 - \tfrac{u_{02}(T_0)}{\tau_0}I_1\Big)\\
& * \Big(-\tfrac{\tau_0 V}{t_{g0}-T_0}\dot{u}_{02}(T_0) + \tfrac{V\tau_0}{t_{g0}-T_0} - \tfrac{\dot{u}_{01}(T_0)}{t_{g0}-T_0}I_2 + \tfrac{\dot{u}_{02}(T_0)}{t_{g0}-T_0}I_1\Big)\\
- \Big(V\dot{u}_{02}(T_0) - V + {} & \tfrac{\dot{u}_{01}(T_0)}{\tau_0}I_2 - \tfrac{\dot{u}_{02}(T_0)}{\tau_0}I_1\Big)\\
& * \Big(-\tfrac{\tau_0 V}{t_{g0}-T_0}u_{02}(T_0) - V\tau_0 - \tfrac{u_{01}(T_0)}{t_{g0}-T_0}I_2 + \tfrac{u_{02}(T_0)}{t_{g0}-T_0}I_1\Big),
\end{aligned}
$$

where we recall that I_k reads

$$
I_k = \int_{t_{g0}}^{T_0} \frac{u_{0k}(s)}{w(s)}\left\{2\ddot{u}_0(s) + \tau_0 F'\left(V + \frac{\dot{u}_0(s)}{\tau_0}\right)\dot{u}_0(s)\right\} ds.
$$

Thus we have

$$
\begin{aligned}
&\det D_{(\tau,t_g)}\Psi(\tau_0, t_{g0}, 0) =\\
&-\tfrac{\tau_0 V^2}{t_{g0}-T_0}u_{02}(T_0)\dot{u}_{02}(T_0) + \tfrac{\tau_0 V^2}{t_{g0}-T_0}u_{02}(T_0) - \tfrac{V}{t_{g0}-T_0}\dot{u}_{01}(T_0)u_{02}(T_0)I_2 + \tfrac{V}{t_{g0}-T_0}u_{02}(T_0)\dot{u}_{02}(T_0)I_1\\
&-\tfrac{\tau_0 V^2}{t_{g0}-T_0}t_{g0}\dot{u}_{02}(T_0) + \tfrac{\tau_0 V^2}{t_{g0}-T_0}t_{g0} - \tfrac{V t_{g0}}{t_{g0}-T_0}\dot{u}_{01}(T_0)I_2 + \tfrac{V t_{g0}}{t_{g0}-T_0}\dot{u}_{02}(T_0)I_1\\
&-\tfrac{V}{t_{g0}-T_0}\dot{u}_{02}(T_0)u_{01}(T_0)I_2 + \tfrac{V}{t_{g0}-T_0}u_{01}(T_0)I_2 - \tfrac{u_{01}(T_0)\dot{u}_{01}(T_0)}{\tau_0(t_{g0}-T_0)}I_2^2 + \tfrac{u_{01}(T_0)\dot{u}_{02}(T_0)}{\tau_0(t_{g0}-T_0)}I_1 I_2\\
&+\tfrac{V}{t_{g0}-T_0}u_{02}(T_0)\dot{u}_{02}(T_0)I_1 - \tfrac{V}{t_{g0}-T_0}u_{02}(T_0)I_1 + \tfrac{u_{02}(T_0)\dot{u}_{01}(T_0)}{\tau_0(t_{g0}-T_0)}I_1 I_2 - \tfrac{u_{02}(T_0)\dot{u}_{02}(T_0)}{\tau_0(t_{g0}-T_0)}I_1^2\\
&-\Big(-\tfrac{\tau_0 V^2}{t_{g0}-T_0}\dot{u}_{02}(T_0)u_{02}(T_0) - \tau_0 V^2\dot{u}_{02}(T_0) - \tfrac{V}{t_{g0}-T_0}\dot{u}_{02}(T_0)u_{01}(T_0)I_2 + \tfrac{V}{t_{g0}-T_0}\dot{u}_{02}(T_0)u_{02}(T_0)I_1\Big)\\
&-\Big(-\tfrac{\tau_0 V^2}{t_{g0}-T_0}u_{02}(T_0) + \tau_0 V^2 + \tfrac{V}{t_{g0}-T_0}u_{01}(T_0)I_2 - \tfrac{V}{t_{g0}-T_0}u_{02}(T_0)I_1\Big)\\
&-\Big(\tfrac{V}{t_{g0}-T_0}\dot{u}_{02}(T_0)u_{02}(T_0)I_1 + V\dot{u}_{02}(T_0)I_1 + \tfrac{u_{01}(T_0)\dot{u}_{02}(T_0)}{\tau_0(t_{g0}-T_0)}I_1 I_2 - \tfrac{u_{02}(T_0)\dot{u}_{02}(T_0)}{\tau_0(t_{g0}-T_0)}I_1^2\Big)\\
&-\Big(\tfrac{V}{t_{g0}-T_0}u_{02}(T_0)\dot{u}_{01}(T_0)I_2 - V\dot{u}_{01}(T_0)I_2 - \tfrac{u_{01}(T_0)\dot{u}_{01}(T_0)}{\tau_0(t_{g0}-T_0)}I_2^2 + \tfrac{u_{02}(T_0)\dot{u}_{01}(T_0)}{\tau_0(t_{g0}-T_0)}I_1 I_2\Big).
\end{aligned}
$$

It remains

$$
\begin{aligned}
&\det D_{(\tau,t_g)}\Psi(\tau_0, t_{g0}, 0)\\
&= -\tfrac{\tau_0 V^2}{t_{g0}-T_0}t_{g0}\dot{u}_{02}(T_0) + \tau_0 V^2\dot{u}_{02}(T_0) - \tau_0 V^2 + \tfrac{\tau_0 V^2 t_{g0}}{t_{g0}-T_0} - \tfrac{V t_{g0}}{t_{g0}-T_0}\dot{u}_{01}(T_0)I_2\\
&\qquad\qquad\qquad + V\dot{u}_{01}(T_0)I_2 + \tfrac{V t_{g0}}{t_{g0}-T_0}\dot{u}_{02}(T_0)I_1 - V\dot{u}_{02}(T_0)I_1,\\
&= -\tfrac{\tau_0 V^2 T_0}{t_{g0}-T_0}\dot{u}_{02}(T_0) + \tfrac{\tau_0 V^2}{t_{g0}-T_0}T_0 - \tfrac{V T_0}{t_{g0}-T_0}\dot{u}_{01}(T_0)I_2 + \tfrac{V T_0}{t_{g0}-T_0}\dot{u}_{02}(T_0)I_1.
\end{aligned}
$$

So we deduce

$$\det D_{(\tau,t_g)}\Psi(\tau_0,t_{g0},0) \neq 0 \Leftrightarrow -\tau_0 V \dot{u}_{02}(T_0) + \tau_0 V + \dot{u}_{02}(T_0)I_1 - \dot{u}_{01}(T_0)I_2 \neq 0. \quad \text{(CC)}$$

If we denote by $\tilde{h} = 2\ddot{u}_0 + \tau_0 F'\left(V + \frac{\dot{u}_0}{\tau_0}\right)\dot{u}_0$, we notice that

$$\frac{1}{\tau_0^2}\dot{u}_{p,\tilde{h}}(T_0) = \dot{u}_{02}(T_0)I_1 - \dot{u}_{01}(T_0)I_2.$$

It follows that (CC) is equivalent to

$$-\tau_0 V \dot{u}_{02}(T_0) + \tau_0 V + \frac{1}{\tau_0^2}\dot{u}_{p,\tilde{h}}(T_0) \neq 0. \quad \text{(CC)}$$

Conclusion Thus, if (CC) is satisfied, then we can solve (4.29) with the Implicit Function Theorem. We then obtain $(\tau, t_g) = (\tau^*(\ell), t_g^*(\ell))$ in a neighbourhood of $(\tau, t_g, \ell) = (\tau_0, t_{g_0}, 0)$ in $\mathbb{R}^3$. And finally we have proved the existence of $u^*(t,\ell) = u_0(t) + u_1^*(\tau(\ell), t_g(\ell), \ell)$ solution to inclusion (2.5) in a neighbourhood of $(u_0, \tau_0, 0)$. So it concludes the proof of Theorem 2.7.

□

Whether the condition (CC) is satisfied or not remains theoretically an open question. However, it is possible to compute numerically the value of (CC). We give here a table of values of (CC) for several sets of parameters V, F_0 and for $\tau_0 = 1$. Computations give that this condition is true except maybe in some isolated values of the set of parameters.

TABLE 4.1 – Values of the condition (CC) for different values of parameters τ_0, V, F_0.

	$\tau_0 = 1$, $V = 0.5$			$\tau_0 = 1$, $V = 0.1$			$\tau_0 = 5$, $V = 0.01$		
F_0	20	50	100	20	50	100	20	50	100
CC	-132	-335	-804	-136	-334	-754	-679	-1687	-3813
$\frac{T_0 - t_{g0}}{T_0}$	$4.7.10^{-2}$	$1.7.10^{-2}$	$8.3.10^{-3}$	$9.9.10^{-4}$	$6.2.10^{-4}$	$3.5.10^{-4}$	$9.9.10^{-4}$	$3.5.10^{-4}$	$1.7.10^{-4}$

Deuxième partie

Modulated waves in the Burridge-Knopoff model, combined with a rate and state friction law

Chapitre 5

Introduction

5.1 The Burridge-Knopoff model with a rate and state friction law

Part I was dedicated to the Burridge-Knopoff model combined with a velocity-weakening friction law. But as said in the Introduction, this 1D-springs model can be combined with many friction laws, experimentally determined. In this second part we are interested in the Burridge-Knopoff model combined with another type of friction law, introduced by Dieterich, Ruina and Rice in rocks mechanics (see for instance [Rui83], [GRRT84]). It is the so-called rate and state friction law. This approach has also been extended by Batista and Carlson for lubricated surface (see [BC98]).

Rate refers to the fact that the force law depends on the instantaneous rate of deformation, and state refers to the fact that the force law depends on the internal state of the system, which incorporates the history dependence. To date, the state variable(s) have been hand crafted, based on physical mechanisms which in some cases have been deduced from experiments or molecular dynamics simulations. In the case of Carlson's work on friction in boundary lubrication, the state variable was associated with the degree of melting in the lubricant. Rate and state laws are inspired by microscopic physics, and allow study of the implications of the microscopic phenomena at larger scales.

The equations of motion are of type

$$\ddot{u}_j + u_j = \ell^2 \left(u_{j+1} - 2u_j + u_{j-1} \right) + F(\dot{u}_j, \theta_j), \quad j \in \mathbb{Z},$$

and are combined with an evolution equation for the state variable θ_j.
We first raise the same question as in Part I.

Problem 1 : *In the rate and state problem, can we expect to find out periodic travelling waves close to the origin ?*

The answer is positive if we are near a critical variety for a four dimension parameter and if the coupling parameter ℓ is close to 0. We will see why below. For that second system, we are also interested in another problem.

Problem 2 : *Can we describe the modulated waves of small amplitude, solutions to the non linear rate and state system, in at least a finite (but long) time interval ?*

These two problems are of very different type. Namely, the first one consists in finding one periodic solution to the system (and so this solution exists for all times). The second problem consists in giving a description of a class of small solutions for at least long time scales.

The answer of Problem 1 has been given by perturbation methods (Lyapounov-Schmidt reduction), and so the result has been obtained for small ℓ. For both the weakening velocity system and rate and state system, we could also use the Centre Manifold theory in the infinite dimensional case to analyse small amplitude solutions to the advance and delay equation (2.5) as done in [IK98, IK00, JS05].
Conversely, we can also use the anticontinuous limit approach for the rate and sate system, since it is smooth. But in this case the result will be local in the parameter space as explained in the following.
The answer of Problem 2 will be given by the so-called Modulation Theory, which can be seen as the counterpart of the Central Manifold Theorem in the case of infinite dimension centre manifold. We have in this second case no restriction on the parameter ℓ.
The aim of part II is the study of Problem 2.
The Modulation Theory is divided into two parts. The purpose of the first part, called Formal Derivation, is to formally derive a so-called amplitude equation, which is the counterpart of the reduced equation on the centre manifold, and whose solutions are supposed to describe the dynamics of small solutions to the original problem. The aim of the second part, called Justification, is to prove the validity of this amplitude equation. That is, to prove that the amplitude equation indeed describes the dynamics of small solutions.
Before giving more details about these two problems, let us introduce the friction law and the equations of motion in the rate and state case.
We recall that the dimensionless equations of motion for the Burridge-Knopoff model are the following

$$\ddot{u}_j + u_j = \ell^2 \left(u_{j+1} - 2u_j + u_{j-1}\right) + F, \quad j \in \mathbb{Z},$$

where F is the friction force. In the rate and state approach, the idea is to account for the ageing of the rocks and then introduce a state variable θ_j. The state variable satisfies an evolution equation. The expression of the friction is determined experimentally. For instance in the paper of Ohmura and Kawamura [OK07], they consider the following friction law

$$F = F(\dot{u}_j, \theta_j) = c + a\ln(1 + V + \dot{u}_j) + b\ln\theta_j,$$

and obtain the following equations of motion (in the moving frame)

$$\begin{cases} \ddot{u}_j + u_j = \ell^2 \left(u_{j+1} - 2u_j + u_{j-1}\right) - \left(c + a\ln(1 + V + \dot{u}_j) + b\ln\theta_j\right), \\ \qquad \dot{\theta}_j = 1 - \theta_j(V + \dot{u}_j), \quad j \in \mathbb{Z}, \end{cases} \tag{5.1}$$

where $u_j(t) \in \mathbb{R}$, $\theta_j(t) \in \mathbb{R}$, $10 \leq a \leq 10^2$, $10^2 \leq b \leq 10^3$ and $10^3 \leq c \leq 10^4$, $V \approx 10^{-8}$ and $\ell \geq 0$ are dimensionless real parameters.
This system has a unique time-space steady solution, which is $u_j = \overline{u}$, $\theta_j = \overline{\theta}$, with $\bar{\theta} = \frac{1}{V}$, $\overline{u} = -c - a\ln(1+V) - b\ln(\bar{\theta})$. Let then introduce $\theta_j = \bar{\theta} + \psi_j$ and $u_j = \overline{u} + v_j$. System (5.1) can thus be written as

$$\begin{cases} \ddot{v}_j + v_j = \ell^2 \left(v_{j+1} - 2v_j + v_{j-1}\right) - \dfrac{a}{1+V}\dot{v}_j + \dfrac{a}{2(1+V)^2}\dot{v}_j^2 - \dfrac{a}{3(1+V)^3}\dot{v}_j^3 \\ \qquad\qquad - bV\psi_j + \dfrac{bV^2}{2}\psi_j^2 - \dfrac{bV^3}{3}\psi_j^3 + R(\dot{v}_j, \psi_j), \\ \dot{\psi}_j = -\dfrac{1}{V}\dot{v}_j - V\psi_j - \psi_j\dot{v}_j, \quad j \in \mathbb{Z}, \end{cases} \tag{5.2}$$

with $R(x,y) = O\left(x^4\right) + O\left(y^4\right)$.

It happens that the justification step is more difficult in the case of a quadratic nonlinearity F rather than in the case of a cubic nonlinearity (we will see why in chapter 8). Thus in part II we will consider a generalized rate and state law, in view of studying the cubic case. From now on, we deal with the following equations of motion

$$\boxed{\begin{cases} \ddot{v}_j + v_j = \ell^2 \left(v_{j+1} - 2v_j + v_{j-1}\right) + a_1\dot{v}_j + b_1\psi_j + F(\dot{v}_j, \psi_j), \\ \qquad \dot{\psi}_j = \alpha_1\dot{v}_j + \beta_1\psi_j + \gamma_{11}\psi_j\dot{v}_j, \quad j \in \mathbb{Z}, \end{cases}} \tag{5.3}$$

where $F(x,y) = a_2x^2 + a_3x^3 + O\left(x^4\right) + b_2y^2 + b_3y^3 + O\left(y^4\right)$, and a_1, b_1, α_1, β_1, γ_{11}, a_2, b_2, a_3, b_3 are real parameters. Following the Ohmura-Kawamura model, we make the following hypothesis on the parameters :

$$\boldsymbol{a_1,\ b_1,\ \alpha_1,\ \beta_1} \ \textbf{\textit{are nonpositive.}} \tag{5.4}$$

We note that system (5.2) is of form (5.3). In (5.3), the friction is given by its general Taylor expansion. Thus if we fix $a_2 = b_2 = \gamma_{11} = 0$, we get a problem with cubic nonlinearity. In chapter 7, we perform the calculations in the general quadratic case, but in chapter 8, for the justification, we will turn off the quadratic part of the nonlinearity to restrict to the cubic case.

5.2 Periodic travelling waves

Let us explain now, why we have already answered to the first problem. Looking for travelling waves, as we have done in Part I in case of the velocity-weakening friction law, we make the ansatz

$$\begin{cases} v_j(t) = v(j - ct), \\ \psi_j(t) = \psi(j - ct). \end{cases}$$

Injecting this into the equations (5.3), we obtain an ODE of order 3 with an advance and delay term.

This rate and state problem is fundamentally not so different from the smoothened weakening-velocity friction law problem of Part I (see Chapter 3). Indeed, the main difference is the order of the ODE system. In Part I, we dealt with a 2-order system, whereas in part II, we deal with a 3-order system. Moreover, the velocity-weakening problem depends on the coupling parameter ℓ and on another single parameter V, whereas the rate and state problem depends on the coupling parameter ℓ, and on a 4-dimension parameter $\Lambda = (a_1, b_1, \alpha_1, \beta_1)$. So for the non coupled problem, in the first case the threshold of instability corresponds to a critical value of the parameter V, whereas in the second case, the threshold of instability corresponds to an algebraic affine variety of $\mathbb{R}^4$. But these problems have the same spectral behaviour : in both cases and for the non coupled problem, we have a Hopf bifurcation. So as we did in Chapter 3, we have a similar theorem as Theorem 2.1. That is, for $(a_1, b_1, \alpha_1, \beta_1)$ close to $(a_1^c, b_1^c, \alpha_1^c, \beta_1^c)$ lying in the critical variety, we have existence of a periodic orbit in one side of the variety. On the the other side, we have stability of the origin.
However, we can not use the Poincaré-Bendixson Theorem to prove then that this periodic orbit exists for all value of the parameter $\Lambda = (a_1, b_1, \alpha_1, \beta_1)$, since we have a 3-dimension system. But we still have local existence of this orbit in a neighbourhood of the critical variety, which enables us to prove persistence for *small* ℓ, and for $(a_1, b_1, \alpha_1, \beta_1)$ close to $(a_1^c, b_1^c, \alpha_1^c, \beta_1^c)$. Indeed, we can apply Theorem 2.7 : this theorem has been proved in a general case of friction law, including the velocity-weakening and rate and state cases. Thus the following holds.

Proposition 1. *Close to the critical variety of the parameter space, for small ℓ, the rate and state problem has a family of periodic orbits in a neighbourhood of the origin, parametrized by the period T, which is a small perturbation of the periodic travelling wave that exists in the non coupled case ($\ell = 0$).*

Now, if we are interesting in proving existence of waves phenomena for all ℓ, this approach is not satisfying, since it is a perturbative approach. To do this, we have to enlarge the class of solutions that we are looking for. That is, instead of considering periodic travelling waves, we will look for **modulated waves**. We will see that this approach as well as imposing no condition on ℓ, will lead us to describe many small solutions of our system near the threshold of instability, but in finite time interval. To explain the idea of this modulated waves approach, let us describe the bifurcation.

5.3 Bifurcation

In system (5.3), we have a stationary uniform state at the origin. The linearized system depends on a 4-dimension parameter $\Lambda = (a_1, b_1, \alpha_1, \beta_1)$. The spectral analysis of this system will be performed in chapter 6. To investigate stability, we look for the real part of the elements of the spectrum. Due to the translation invariance, the spectrum is obtained by substituting $v_j = e^{iqj+\lambda t}\widehat{v}$ and $\varphi_j = e^{iqj+\lambda t}\widehat{\varphi}$ into the equations. It gives (see Proposition 6.2) that the spectrum is a continuous set and consists in the roots of

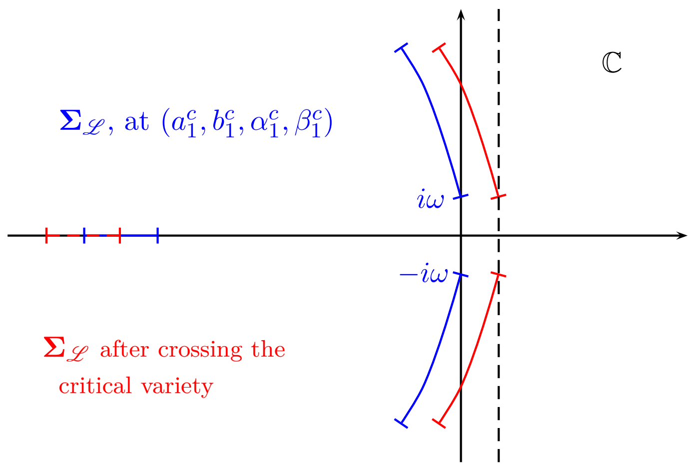


FIGURE 5.1 : Spectrum of $\mathscr{L}$

a polynomial of degree 3

$$P(X, q) = X^3 - (a_1 + \beta_1)X^2 + (a_1\beta_1 - \alpha_1 b_1 + 1 + 4\ell^2 \sin^2(q/2))X - \beta_1(1 + 4\ell^2 \sin^2(q/2),$$

depending on $\Lambda = (a_1, b_1, \alpha_1, \beta_1)$ and parametrized by the wavenumber q in $\mathbb{R}$. We find that there exists a region in $\mathbb{R}^4$ for this parameter, delimited by an affine variety, such that in this region, the basic state is stable. Let us then consider that we cross this critical variety at a point $(a_1^c, b_1^c, \alpha_1^c, \beta_1^c)$. We set $\Lambda = \Lambda(\mu) = (a_1(\mu), b_1(\mu), \alpha_1(\mu), \beta_1(\mu))$ with $\mu = \rho\varepsilon^2$, a small parameter and $\rho = \pm 1$, such that we cross the boundary for $\mu = 0 : \Lambda(0) = (a_1^c, b_1^c, \alpha_1^c, \beta_1^c)$ and $\Lambda(\mu) \in \mathscr{R}^+$ or $\mathscr{R}^-$ depending on the sign of μ. Then on one side of the variety, the system becomes unstable (cf Theorem 6.6). More precisely, at $\mu = 0$, the polynomial $P(X, q)$ has a purely imaginary root for $q = 0$ and all other roots have negative real parts. And if we cross the boundary, there is a wide band of wavenumbers q, for which the roots of the polynomial $P(X, q)$ have positive real parts. In other words, it means that a whole band of modes q becomes unstable, and the mode $q = 0$ is the first to become unstable. So a bifurcation occurs at any point $\Lambda^c = (a_1^c, b_1^c, \alpha_1^c, \beta_1^c)$ of this boundary. In a neighbourhood of such a point, the shape of the spectrum is given by figure 5.1. So clearly, looking at this figure, we see that we do not have a Hopf bifurcation because we have no spectral gap.

5.4 The modulated waves approach

As announced, we now consider instead of the travelling wave ansatz, a modulated wave ansatz. That means that we allow the amplitude of the periodic wave to vary slowly with space and time. The starting point of this approach is the following.

If $\lambda(\mu, q)$ is an eigenvalue of the linear operator $\mathscr{L}(\mu, iq)$ and $\Phi_{\mu,q}$ an associated eigenvector, then $V = (e^{\lambda(\mu,q)t+iqj})_j\Phi_{\mu,q}$ is a solution to the linear problem. Let us then suppose that have a single pair of eigenvalues that crosses the imaginary axis at $\mu = 0$. So we have at $\mu = 0$, for a certain q_0, $\lambda(0, q_0) = i\omega$. Then following the notations of [Mie02] (page 766), we write for $\lambda(\mu, q)$ an eigenvalue of $\mathscr{L}(\mu, iq)$ with largest real part,

$$\lambda(\mu, q) = \lambda_0(\mu)(q - q_0) + \lambda_1(\mu)(q - q_0)^2 + \lambda_2(\mu)(q - q_0)^2 + 0(|q - q_0|^3).$$

We thus have $\mathrm{Re}\,\lambda_0(0) = 0$, $\frac{d\mathrm{Re}\,\lambda_0}{d\mu}(0) > 0$ (i.e. we cross the imaginary axis at $\mu = 0$ from left to right). Suppose also that $\mathrm{Re}\,\lambda_1(0) = 0$.

Then for (μ, q) close to $(0, q_0)$ we have

$$\mu = \rho\varepsilon^2, \quad \rho = \pm 1,$$
$$q = q_0 + \varepsilon\kappa,$$

and it follows that $\lambda(\mu, q)$ can be written in the form

$$\lambda(\mu, q) = i\omega + \varepsilon^2\rho\lambda_{0,1} - \varepsilon i C_{gr}\kappa - \varepsilon^2\Lambda_0^2 + O(\varepsilon^3),$$

with $C_{gr} = -i\lambda_1(0)$ and $\Lambda_0 = \Lambda(0)$.

Thus the above solutions of the linear problem can be written as

$$e^{\lambda(\mu,q)+iqj} = e^{(\rho\lambda_{0,1}-\Lambda_0\kappa^2)\varepsilon^2 t+i\kappa\varepsilon(j-C_{gr})t+O(\varepsilon^3)}\underbrace{e^{i(\omega t+q_0 j)\Phi_{\varepsilon^2\rho,q_0+\varepsilon\kappa}}}_{\approx p(t,j)=e^{i(\omega t+q_0 j)\Phi_{0,q_0}}}.$$

At $\mu = 0$, $p(t, x)$ is the only mode that is not damped. We notice that the amplitude depends on the slow time variable $\varepsilon^2 t$ and on the slow space variable $\xi = \varepsilon(j - C_{gr}t)$. Thus we expect that, considering

$$V^A(t) = A(\tau, \xi)p(t, x) + c.c.,$$

there exist solutions to the original (nonlinear) system that are close to V^A for small $\mu > 0$.

Lastly, as ε^2 is close to zero, we expect these modulated waves to be of small amplitude (since they appear from the bifurcation). We thus make a scaling and suppose that the amplitude is of order $O(\varepsilon)$ (to make a parallel with the Hopf bifurcation, the orbits obtained in this case are of order $\sqrt{|\mu|}$).

Thus, the purpose is to prove existence of solutions to the system (5.3) that are close (in a sense that will be specified in Chapter 8) to modulated waves of the form

$$V^A(t) = \begin{pmatrix} v^A(t) \\ \dot{v}^A(t) \\ \psi^B(t) \end{pmatrix},$$

with

$$\begin{cases} v_j^A(t) = \varepsilon A\left(\varepsilon^2 t, \varepsilon(j - C_{gr}t\right) e^{i\omega t + iq_0 j} + c.c., \\ \psi_j^B(t) = \varepsilon B\left(\varepsilon^2 t, \varepsilon(j - C_{gr}t\right) e^{i\omega t + iq_0 j} + c.c, \end{cases} \tag{5.5}$$

where c.c. stands for complex conjugated, since we are looking for real solutions. We will see that in our case, $q_0 = 0$ and $C_{gr} = 0$.

The following is divided into two steps :

• **Step 1 : Formal derivation of the Ginzburg Landau equation.** It is the investigation for a necessary condition. If a solution close to the ansatz (5.5), exists then substituting it into the system (5.3) and identifying the powers of ε, leads to a so-called amplitude equation. That is, if such solutions exist, then the amplitudes A and B have to satisfy a certain partial differential equation, called **amplitude equation**. Chapter 7 is dedicated to the formal derivation of this equation. We obtain for the amplitude equation, a complex Ginzburg-Landau equation of the form

$$\partial_\tau A = (c + id)A + (\nu + i\alpha)\partial_{\xi\xi}A + (a + ib)|A|^2 A, \quad A(\tau, \xi) \in \mathbb{C}.$$

The accurate result is given in Theorem 7.2, page 95.

• **Step 2 : Validity of the Ginzburg-Landau equation as an amplitude equation (Justification).** We want to prove the validity of this amplitude equation, in the following sense : does this equation really describe the dynamics of small solutions to our original system (5.3) ? We will answer to this in the following sense :

Theorem 1. *Given an amplitude A solution to the Ginzburg-Landau equation, every solution to the problem* (5.3) *with initial condition $O(\varepsilon^{\frac{3}{2}})$-close to $A(0)$, remains $O(\varepsilon^{\frac{3}{2}})$-close to the formal approximation* (5.5) *on a time scale of order $O(\frac{1}{\varepsilon^2})$.*

The accurate result is given by Theorem 8.1, page 103.

Chapitre 6

Spectral Analysis

The object of this chapter is to make the spectral analysis of our problem and to find out a bifurcation. So we first introduce the linearized operator at the basic solution 0 and look for its spectrum. Then we analyze the sign of the real parts to investigate stability at 0.

6.1 Notations

We will denote by $\Delta_d\, v = v_{j+1} - 2v_j + v_{j-1}$ a discretization of the Laplacian operator. Let us also write system (5.3) as a first order differential equation in the Hilbert space $\mathcal{Y} = \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$:

$$\frac{dV}{dt} = \mathscr{L}V + \mathscr{N}(V), \tag{6.1}$$

where $V = \begin{pmatrix} v \\ \dot{v} \\ \psi \end{pmatrix} \in \mathcal{Y}$ and $v = (v_j)_{j\in\mathbb{Z}}$, $\dot{v} = (\dot{v}_j)_{j\in\mathbb{Z}}$, $\psi = (\psi_j)_{j\in\mathbb{Z}}$ are elements of $\ell^2(\mathbb{Z})$. The linear operator $\mathscr{L}$ acting on $\mathcal{Y}$ is given by

$$\mathscr{L} = \begin{pmatrix} 0 & 1 & 0 \\ -1 - \ell^2\Delta_d & a_1 & b_1 \\ 0 & \alpha_1 & \beta_1 \end{pmatrix}, \tag{6.2}$$

and the nonlinear part is given by

$$\mathscr{N}(U) = \begin{pmatrix} 0 \\ N(U) \\ M(U) \end{pmatrix} \quad \text{for} \quad U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathcal{Y}, \tag{6.3}$$

where

$$\begin{cases} N(U) = N_1(u_2) + N_2(u_3), \\ M(U) = \gamma_{11} u_2 u_3, \end{cases} \tag{6.4}$$

and

$$\begin{cases} N_1(u_2) = a_2u_2^2 + a_3u_2^3 + \widehat{N_1}(u_2), & \widehat{N_1}(u_2) = O(u_2^4), \\ N_2(u_3) = b_2u_3^2 + b_3u_3^3 + \widehat{N_2}(u_3), & \widehat{N_2}(u_3) = O(u_3^4). \end{cases}$$

We clearly have for $U \in \mathcal{Y}$, $N(U) \in \ell^2(\mathbb{Z})$ and $M(U) \in \ell^2(\mathbb{Z})$. For simplicity, we again denote by $\widehat{N_1}$ and $\widehat{N_2}$ the functions such that $\left(\widehat{N_1}(u_2)\right)_j = \widehat{N_1}(u_{2,j})$ and $\left(\widehat{N_2}(u_3)\right)_j = \widehat{N_2}(u_{3,j})$. We suppose in the following that $\widehat{N}_1$ *and* $\widehat{N}_2$ *are smooth functions* : $\widehat{N}_1, \widehat{N}_2 \in C^5(\mathbb{R}, \mathbb{R})$.

6.2 Spectrum and resolvent of $\mathscr{L}$

We first note that we have the following simple result.

Lemma 6.1. *The operator $\mathscr{L}$ is a bounded operator from $\mathcal{Y}$ into $\mathcal{Y}$.*

We thus deduce that $\mathscr{L}$ has a **non-empty compact spectrum**. We denote respectively by $\Sigma_{\mathscr{L}}$ and $\rho(\mathscr{L})$, the spectrum and the resolvent of $\mathscr{L}$. Let us prove the following result giving $\Sigma_{\mathscr{L}}$ as roots of a polynomial P.

Proposition 6.2

The spectrum of $\mathscr{L}$ is the set

$$\Sigma_{\mathscr{L}} := \{\lambda \in \mathbb{C} / \text{ there exists } q \in [0, \pi],\ P(\lambda, q) = 0\},$$

where $P(X, q)$ is a polynomial of degree 3 in X depending on the wavenumber $q \in [0, \pi]$, defined by

$$P(X, q) = X^3 - (a_1 + \beta_1)X^2 + (a_1\beta_1 - \alpha_1 b_1 + \Omega_q)X - \beta_1\Omega_q, \tag{6.5}$$

where $\Omega_q = 1 + 4\ell^2 \sin^2(q/2)$.

Thus $\mathscr{L}$ has a continuous spectrum parametrized by q, which is the set of the three roots of the polynomials $P(X, q)$ when q lies in $[0, \pi]$.

Proof: As $\mathscr{L}$ is a linear operator in an infinite dimensional Banach space, the spectrum of $\mathscr{L}$ are the values of $\lambda \in \mathbb{C}$ such that $\mathscr{L} - \lambda Id$ is non injective or non surjective on $\mathcal{Y}$.

So let $F = (f_1, f_2, f_3) \in \mathcal{Y}$ and let us solve the equation $\mathscr{L}U - \lambda U = F$, of unknown $U = (u_1, u_2, u_3)$. We obtain

$$\mathscr{L}U - \lambda U = F \Longleftrightarrow \begin{cases} -\lambda u_1 + u_2 = f_1, \\ (-1 + \ell^2\Delta_d)u_1 + (a_1 - \lambda)u_2 + b_1u_3 = f_2, \\ \alpha_1 u_2 + (\beta_1 - \lambda)u_3 = f_3, \end{cases}$$

$$\Longleftrightarrow \begin{cases} u_2 = f_1 + \lambda u_1, \\ u_3 = \dfrac{f_3 - \alpha_1 f_1 - \alpha_1 \lambda u_1}{\beta_1 - \lambda}, \\ \left[-1 + \lambda(a_1 - \lambda) + \dfrac{b_1\alpha_1}{\beta_1 - \lambda}\lambda + \ell^2\Delta_d\right] u_1 = \left[\lambda - a_1 + \dfrac{b_1\alpha_1}{\beta_1 - \lambda}\right] f_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + f_2 - \dfrac{b_1}{\beta_1 - \lambda} f_3. \end{cases}$$

It is easier to solve the third equation after doing a kind of Fourier Transform that we describe now. Let us denote by $\mathcal{F}$ the Fourier isometry

$$\begin{array}{rcl} \mathcal{F} : L^2(T) & \longrightarrow & \ell^2(\mathbb{Z}) \\ f & \longmapsto & (\hat{f}(n))_{n\in\mathbb{Z}}, \end{array}$$

where $(\hat{f}(n))_{n\in\mathbb{Z}} = (C_n(f))_{n\in\mathbb{Z}}$ denotes the sequence of Fourier coefficients of a 2π-periodic function f in L^2 (T denotes the torus $\mathbb{R}/2\pi$). By Plancherel's Theorem we have $\|f\|_{L^2(T)} = \|\mathcal{F}[f]\|_{\ell^2(\mathbb{Z})}$.

We denote by $\mathcal{F}^{-1} : \ell^2(\mathbb{Z}) \longrightarrow L^2(T)$ its inverse bijection : given a sequence a in $\ell^2(\mathbb{Z})$, $\mathcal{F}^{-1}[a]$ is the unique function in $L^2(T)$ such that a is the sequence of its Fourier coefficients. We thus have $\|a\|_{\ell^2(\mathbb{Z})} = \left\|\mathcal{F}^{-1}[a]\right\|_{L^2(T)}$.

Finally, we also denote by $\boldsymbol{\mathcal{F}}$ the operator $\mathcal{Y} \longrightarrow \mathcal{Z}$, such that

$$\boldsymbol{\mathcal{F}}[U] = \begin{pmatrix} \mathcal{F}[u_1] \\ \mathcal{F}[u_2] \\ \mathcal{F}[u_3] \end{pmatrix},$$

and $\boldsymbol{\mathcal{F}^{-1}}$ its inverse bijection $\mathcal{Z} \longrightarrow \mathcal{Y}$.

To apply operator $\mathcal{F}^{-1}$ to the third equation, we need the following lemma.

Lemma 6.3. *Given $a \in \ell^2(\mathbb{Z})$, we have $\mathcal{F}^{-1}[\Delta_d\ a] = \check{L}(a) \in L^2(T)$, where $\check{L}(a)(q) = -4\sin^2(q/2)\mathcal{F}^{-1}[a]$.*

<u>*Proof:*</u> We denote by $\check{a} := \mathcal{F}^{-1}[a]$, $\check{L}(a) := \mathcal{F}^{-1}[\Delta_d\ a]$ and $\check{g}(a) := -4\sin^2(q/2)\check{a}$. We want to show that $\check{L}(a) = \check{g}(a)$, which is equivalent to $\mathcal{F}[\check{L}(a)] = \mathcal{F}[\check{g}(a)]$ or again to $\Delta_d\ a = \mathcal{F}[\check{g}(a)]$. Let us then calculate $\mathcal{F}[\check{g}(a)]$:

$$\begin{array}{rcl} C_n(\check{g}(a)) & = & \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} -4\sin^2(\frac{q}{2})e^{-inq}\check{a}(q)dq, \\ & = & \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} -2(1-\cos q)e^{-inq}\check{a}(q)dq, \\ & = & \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} -2e^{-inq}\check{a}(q)dq + \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(e^{iq}+e^{-iq})e^{-inq}\check{a}(q)dq, \\ & = & -2C_n(\check{a}) + C_{n+1}(\check{a}) + C_{n-1}(\check{a}), \\ & = & -2a_n + a_{n+1} + a_{n-1}. \end{array}$$

Thus $\mathcal{F}[\check{g}(a)] = \Delta_d\, a$.

$\square$

Let us then apply $\mathcal{F}^{-1}$ to our system of equations. It gives for $\lambda \neq \beta_1$

$$\mathscr{L}U - \lambda U = F$$

$$\Longleftrightarrow$$

$$(*)\left\{\begin{aligned} \check{u}_2 =& \check{f}_1 + \lambda \check{u}_1, \\ \check{u}_3 =& \frac{f_3 - \alpha_1 \check{f}_1 - \alpha_1 \lambda \check{u}_1}{\beta_1 - \lambda}, \\ P(\lambda, q)\check{u}_1 =& (\beta_1 - \lambda)\check{f}_2 + [b_1\alpha_1 - (\beta_1 - \lambda)(a_1 - \lambda)]\, \check{f}_1 - b_1 \check{f}_3, \end{aligned}\right.$$

where we use the notation $\check{a} = \mathcal{F}^{-1}[a]$ for $a \in \ell^2(\mathbb{Z})$. The equations in $(*)$ are to be taken in $L^2(T)$. The polynomial P is given by

$$P(\lambda, q) = \lambda^3 - (a_1 + \beta_1)\lambda^2 + (a_1\beta_1 - \alpha_1 b_1 + \Omega_q)\lambda - \beta_1\Omega_q, \quad \Omega_q = 1 + 4\ell^2 \sin^2(\frac{q}{2}).$$

First, let us note that if $(*)$ has a solution in $\mathscr{Z} = L^2(T) \times L^2(T) \times L^2(T)$, then it is necessary unique. We deduce that there is no eigenvalue in the spectrum. And so the spectrum is the set of $\lambda \in \mathbb{C}$, for which the operator $\mathscr{L} - \lambda Id$ is *non surjective.*

We denote by $\Sigma_q := \{\lambda \in \mathbb{C}/\ P(\lambda, q) = 0\}$. Thus Σ_q is the roots of $P(X, q)$ for fixed $q \in [0, 2\pi]$. The polynomial $P(\lambda, q)$ can be written under the form $P(\lambda, q) = m_1(\lambda)\sin^2(q/2) + m_2(\lambda)$, and so it depends on parameter q only through $\sin^2(q/2)$. So we can restrict q to $[0, \pi]$. We then have $\Sigma_{\mathscr{L}} = \bigcup\limits_{q \in [0,\pi]} \Sigma_q$.

Given $(*)$, λ is in the resolvent set of $\mathscr{L}$ if and only if $\check{u}_1, \check{u}_2, \check{u}_3$ are in $L^2_q(T)$ for any $\check{f}_1$, $\check{f}_2$, $\check{f}_3 \in L^2_q(T)$.

***Case 1** : $\lambda \notin \Sigma_{\mathscr{L}} \cup \{\beta_1\}$.*

In this case, we have for all $q \in [0, \pi]$, $P(\lambda, q) \neq 0$. And $\check{u}_1$ is given by the formula

$$\check{u}_1(q) = \frac{[b_1\alpha_1 - (\beta_1 - \lambda)(a_1 - \lambda)]\check{f}_1 + (\beta_1 - \lambda)\check{f}_2 - b_1\check{f}_3}{P(\lambda, q)}.$$

$P(\lambda, .)$ is a continuous function on the compact $[0, \pi]$, that is never equal to 0. We deduce that $\dfrac{1}{P(\lambda, q)}$ is bounded. In particular, there exists $M > 0$ such that $\left|\dfrac{1}{P(\lambda, q)}\right| \leq M$ for all $q \in [0, \pi]$. Hence it holds

$$\begin{aligned} \|\check{u}_1\|_{L^2_q} &\leq C\left(\left\|\frac{\check{f}_1}{P(\lambda, .)}\right\|_{L^2_q} + \left\|\frac{\check{f}_2}{P(\lambda, .)}\right\|_{L^2_q} + \left\|\frac{\check{f}_3}{P(\lambda, .)}\right\|_{L^2_q}\right), \\ &\leq CM\left(\left\|\check{f}_1\right\|_{L^2_q} + \left\|\check{f}_2\right\|_{L^2_q} + \left\|\check{f}_3\right\|_{L^2_q}\right) < \infty. \end{aligned}$$

Thus, $\check{u}_1 \in L^2_q(T)$. It follows from $(*)$ that $\check{u}_2$ and $\check{u}_3$ are also in $L^2_q(T)$. So $\check{V}$ is in $\mathcal{Z} = L^2_q(T) \times L^2_q(T) \times L^2_q(T)$ and λ is in the resolvent set of $\mathscr{L}$.

Case 2 *:* $\lambda = \beta_1$.

We have

$$\mathscr{L}U - \lambda U = F \Longleftrightarrow \begin{cases} f_1 = -\beta_1 u_1 + u_2, \\ f_2 = \alpha_1 u_2, \\ f_3 = -u_1 + \Delta_d\, u_1 + (a_1 - \beta_1)u_2 + b_1 u_3. \end{cases}$$

And thus for $F \in \mathcal{Y}$, we have since $\beta_1, \alpha_1, b_1 \neq 0$,

$$\begin{cases} \beta_1 u_1 = \dfrac{1}{\alpha_1} f_3 - f_1 \in \ell^2(\mathbb{Z}), \\ \alpha_1 u_2 = f_3 \in \ell^2(\mathbb{Z}), \\ b_1 u_3 = f_2 + (1 + \Delta_d)u_1 - (a_1 - \beta_1)u_2 \in \ell^2(\mathbb{Z}). \end{cases}$$

We deduce that $\beta_1 \in \rho(\mathscr{L})$.

Case 3 *:* $\lambda \in \Sigma_{\mathscr{L}}$.

Then there exists a unique $q_0 \in [0, \pi]$ such that $P(\lambda, q_0) = 0$. Hence $\check{u}_1(q)$ has a singularity for $q = q_0$. We have to look at the nature of this singularity to see whether $|\check{u}_1|^2$ is integrable or not in $[0, 2\pi]$ for any $\check{f}_1$, $\check{f}_2$, $\check{f}_3 \in L^2_q(T)$. We recall the formula

$$\check{u}_1(q) = \frac{[b_1\alpha_1 - (\beta_1 - \lambda)(a_1 - \lambda)]\check{f}_1 + (\beta_1 - \lambda)\check{f}_2 - b_1\check{f}_3}{P(\lambda, q)}.$$

We again use the notation $P(\lambda, q) = m_1(\lambda)\sin^2(q/2) + m_2(\lambda)$ with $m_2(\lambda) = P(\lambda, 0) = \lambda^3 - (a_1 + \beta_1)\lambda^2 + (1 + a_1\beta_1 - b_1\alpha_1)\lambda - \beta_1$ and $m_1(\lambda) = 4\ell^2(\lambda - \beta_1)$.

We note that $m_1(\lambda) \neq 0$ since $\lambda \neq \beta_1$. If $q_0 \neq 0$ then in a neighbourhood of the singularity q_0, we have $P(\lambda, q) \sim m_1(\lambda)\sin(q_0/2)\cos(q_0/2)(q - q_0)$. And so $\check{u}_1$ is not in $L^2_q(T)$. If $q_0 = 0$ then $P(\lambda, q_0) = P(\lambda, 0) = m_2(\lambda) = 0$ and $P(\lambda, q) = m_1(\lambda)\sin^2(q/2) \sim_0 \dfrac{m_1(\lambda)}{4} q^2$. Thus $\check{u}_1$ is not in $L^2_q(T)$. So in the case $\lambda \in \Sigma_{\mathscr{L}}$, $\mathscr{L} - \lambda Id$ is not surjective.

Conclusion. The spectrum of $\mathscr{L}$ is the roots of polynomials P for q lying in $[0\,, \pi]$.

□

Expression of the resolvent in the space $\mathcal{Z}$. Given system $(*)$, we write here the expression of the resolvent of $\mathscr{L}$ for ζ in the resolvent set $\rho(\mathscr{L}) = \mathbb{C} \setminus \Sigma_{\mathscr{L}}$. It will be useful in chapter 8 to get an estimate of $e^{t\mathscr{L}}$.

Let us denote by $R(\zeta, \mathscr{L}) = (\mathscr{L} - \zeta Id)^{-1}$, the resolvent for $\zeta \in \rho(\mathscr{L})$. So we have

$$(\mathscr{L} - \zeta Id)U = F \Longleftrightarrow V = R(\zeta, \mathscr{L})F.$$

We do not have an explicit expression of U, but we have an explicit one of $\check{U}$. More precisely, if we denote by $\check{R}(\zeta, q, \mathscr{L})$ the symbol of the resolvent, we have

$$\begin{aligned} U = R(\zeta, \mathscr{L})F \quad &\Longleftrightarrow \quad \mathcal{F}^{-1}[u] = \mathcal{F}^{-1}\left[R(\zeta, \mathscr{L})F\right] \\ &\Longleftrightarrow \quad \check{U} = \check{R}(\zeta, q, \mathscr{L})\check{F}. \end{aligned}$$

Given the formulas $(*)$ we can write

$$\check{R}(\zeta, q, \mathscr{L}) = \Big(\check{r}_{ij}(\zeta, q)\Big)_{1 \leq i,j \leq 3}, \tag{6.6}$$

with

$$\begin{aligned}
\check{r}_{11}(\zeta, q) &= \frac{b_1\alpha_1 - (\beta_1 - \zeta)(a_1 - \zeta)}{P(\zeta, q)}, \\
\check{r}_{12}(\zeta, q) &= \frac{\beta_1 - \zeta}{P(\zeta, q)}, \\
\check{r}_{13}(\zeta, q) &= -\frac{b_1}{P(\zeta, q)}, \\
\check{r}_{21}(\zeta, q) &= 1 + \zeta\check{r}_{11}(\zeta, q), \\
\check{r}_{22}(\zeta, q) &= \zeta\check{r}_{12}(\zeta, q), \\
\check{r}_{23}(\zeta, q) &= \zeta\check{r}_{13}(\zeta, q), \\
\check{r}_{31}(\zeta, q) &= -\alpha_1\frac{1 + \zeta\check{r}_{11}(\zeta, q)}{\beta_1 - \zeta}, \\
\check{r}_{32}(\zeta, q) &= -\alpha_1\frac{\zeta\check{r}_{12}(\zeta, q)}{\beta_1 - \zeta}, \\
\check{r}_{33}(\zeta, q) &= \frac{1 - \alpha_1\zeta\check{r}_{13}(\zeta, q)}{\beta_1 - \zeta}.
\end{aligned}$$

Remark 6.4. *$\zeta = \beta_1$ may appear in some of the formulas as a singularity. But it is not the case, since as we saw, $\beta_1 \in \rho(\mathscr{L})$. In these formulas, β_1 is also a zero of the numerator.*

6.3 Shape of the spectrum and bifurcation

Up to now, we have seen that the spectrum of $\mathscr{L}$ is given by the roots of a continuously q-parametrized polynomial P given by (6.5). We now investigate more precisely the spectrum and look for a possible bifurcation at some critical value of the parameter $\Lambda = (a_1, b_1, \alpha_1, \beta_1)$ lying in $\mathbb{R}_*^{-4}$.

Let us prove the following proposition. We will then deduce the existence of a bifurcation in our system.

Proposition 6.5

We denote by $\mathscr{R}^-$ and $\mathscr{R}^+$ the regions of $\mathbb{R}^4$ given by

$$\mathscr{R}^- = \left\{(a_1, b_1, \alpha_1, \beta_1) \in \mathbb{R}_*^{-4} \mid (a_1\beta_1 - \alpha_1 b_1)(a_1 + \beta_1) + a_1 < 0\right\},$$

$$\mathscr{R}^+ = \left\{(a_1, b_1, \alpha_1, \beta_1) \in \mathbb{R}_*^{-4} \mid (a_1\beta_1 - \alpha_1 b_1)(a_1 + \beta_1) + a_1 > 0\right\},$$

and by $\mathscr{R}^c$ the boundary

$$\mathscr{R}^c = \left\{(a_1, b_1, \alpha_1, \beta_1) \in \mathbb{R}_*^{-4} \mid (a_1\beta_1 - \alpha_1 b_1)(a_1 + \beta_1) + a_1 = 0\right\}.$$

We also denote by $\lambda^R(q)$, $\lambda^+(q)$, $\lambda^-(q)$ the roots of $P(\lambda, q)$ for fixed $q \in [0, \pi]$.

Then, for every $\Lambda = (a_1, b_1, \alpha_1, \beta_1) \in \mathbb{R}_*^{-4}$ and every q in $[0\,, \pi]$, $P(X, q)$ has one real nonpositive root $\lambda^R(q)$. Concerning the two other roots, it holds :

- *if Λ belongs to the region $\mathscr{R}^-$, then for every q in $[0\,, \pi]$, $\lambda^\pm(q)$ have nonpositive real parts.*
- *if Λ belongs to the boundary of $\mathscr{R}^-$, $\mathscr{R}^c$, then for every q in $[0\,, \pi]$, $\lambda^\pm(q)$ have nonpositive real parts except for $q = 0$, where we have a pair of purely imaginary roots.*
- *if Λ belongs to $\mathscr{R}^+$, then there exists a wide band of modes $[0\,, q_1]$ for which the polynomial $P(X, q)$ has positive real part roots.*

Proof: For fixed $q \in [0, \pi]$, $P(X, q)$ is a polynomial of degree 3. So it has necessary a real root $\lambda^R(q)$. Moreover under hypothesis (5.4) it holds $P(\beta_1, q) = -b_1\alpha_1\beta_1 > 0$. Thus $\lambda^R(q) < \beta_1 < 0$.

To investigate stability we have to find the sign of the real part of the roots of $P(X, q)$. The origin is stable if the spectrum lies in the half plane $\mathbf{Re}\ z < 0$, so if for all q, the roots of $P(X, q)$ have nonpositive real parts. Thus we want to determine whether the polynomial $P(X, q)$ is a Hurwitz polynomial or not, which means exactly that the roots have nonpositive real part. The Routh-Hurwitz criterion gives a necessary and sufficient condition to be a Hurwitz polynomial (see [Der57]). If P is a polynomial of degree 3, denoted by $P(X) = c_3X^3 + c_2X^2 + c_1X + c_0$ with $c_3 > 0$, this criterion says

$$P \text{ is Hurwitz} \iff \begin{cases} \text{for } j = 1 \ldots 3,\ c_j > 0 \ \boldsymbol{(RH1)}, \\ \text{and} \\ c_1c_2 - c_0c_3 > 0 \ \boldsymbol{(RH2)}. \end{cases}$$

Let us investigate for which values of Λ, $P(X, q)$ satisfies conditions **(RH1)** and **(RH2) for all q**. In the following, we denote by $\boldsymbol{\Omega_q = 1 + 4\ell^2 \sin^2(\frac{q}{2}) > 0}$.

• ***Condition (RH1)*** *:* In our case we have $c_3 = 1 > 0$. Under hypothesis (5.4), we also have $c_2 = -(a_1 + \beta_1) > 0$ and $c_0 = -\beta_1\Omega_q > 0$. Thus it remains to satisfy $c_1 > 0$. It gives the first condition

$$a_1\beta_1 - \alpha_1 b_1 + \Omega_q > 0, \quad \text{for all } q \in [0, \pi], \tag{6.7}$$

which is equivalent to

$$a_1\beta_1 - \alpha_1 b_1 + 1 > 0. \tag{6.8}$$

• *Condition **(RH2)** :* We have

$$\begin{aligned} c_1c_2 - c_0c_3 > 0 \quad &\Longleftrightarrow \quad -(a_1+\beta_1)(a_1\beta_1 - \alpha_1 b_1 + \Omega_q) + \beta_1\Omega_q > 0, \\ &\Longleftrightarrow \quad \Omega_q > (a_1+\beta_1)\left(\frac{\alpha_1 b_1}{a_1} - \beta_1\right), \\ &\Longleftrightarrow \quad a_1\beta_1 - \alpha_1 b_1 + \Omega_q > \frac{\beta_1}{a_1}(\alpha_1 b_1 - a_1\beta_1), \end{aligned} \tag{6.9}$$

which is equivalent to

$$a_1\beta_1 - \alpha_1 b_1 + 1 > -\frac{\beta_1}{a_1}(a_1\beta_1 - \alpha_1 b_1). \tag{6.10}$$

If $a_1\beta_1 - \alpha_1 b_1 > 0$, both inequalities (6.8) and (6.10) are satisfied. If $a_1\beta_1 - \alpha_1 b_1 < 0$ then (6.10) implies (6.8). Thus we obtain that $P(X, q)$ has nonpositive real part roots for all q if and only if Λ satisfies

$$(a_1\beta_1 - \alpha_1 b_1)\left(1 + \frac{\beta_1}{a_1}\right) + 1 > 0.$$

That is, if and only if Λ lies in $\mathscr{R}^-$, and so in this case the origin is a stable uniform steady state. Let us look at what happens outside this region.

- If $\Lambda \in \mathscr{R}^c$, then condition (6.7) is satisfied for all q and (6.9) simply writes in this case $4\ell^2\sin^2(\frac{q}{2}) > 0$. So the mode $q = 0$ is critical, since $P(X, 0)$ is not a Hurwitz polynomial.
- If $\Lambda \in \mathscr{R}^+$, then $a_1\beta_1 - \alpha_1 b_1 + 1 < \frac{\beta_1}{a_1}(\alpha_1 b_1 - \beta_1 a_1)$. It follows that there exists q_1 in $[0\,, \pi]$, such that for every q in $[0\,, q_1]$ we have again

  $$a_1\beta_1 - \alpha_1 b_1 + 1 + 4\ell^2\sin^2(\frac{q}{2}) < \frac{\beta_1}{a_1}(\alpha_1 b_1 - \beta_1 a_1).$$

  Thus for all q in $[0\,, q_1]$, the polynomial $P(X, q)$ has positive real parts. And thus all these modes are not stable.

Conclusion : $\mathscr{R}^c$ corresponds to the threshold of instability. And by crossing $\mathscr{R}^c$, a wide band of modes turns unstable.

□

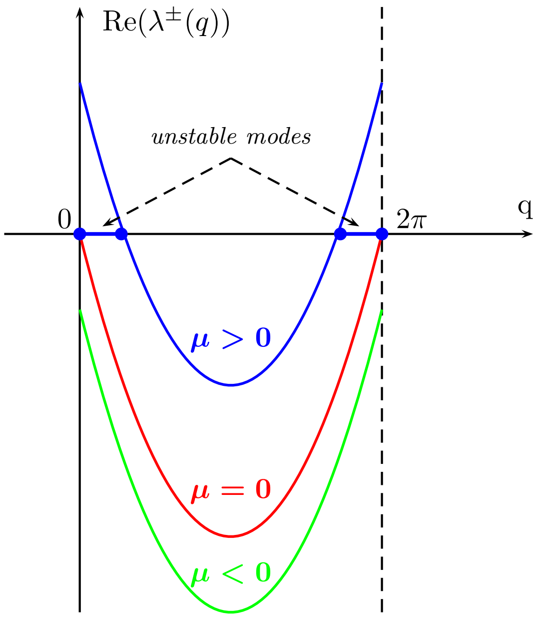


FIGURE 6.1 : Evolution of the real part of the pair of complex eigenvalue, w.r.t. μ.

In the following, we will consider Λ close to some point $\Lambda^c = (a_1^c, b_1^c, \alpha_1^c, \beta_1^c)$, lying in $\mathscr{R}^c$. For this purpose, we introduce a small parameter μ, and write $\Lambda = \Lambda(\mu)$, with

$$\begin{array}{rcl} a_1(\mu) & = & a_1^c + a_{11}\mu + o(\mu), \\ b_1(\mu) & = & b_1^c + b_{11}\mu + o(\mu), \\ \alpha_1(\mu) & = & \alpha_1^c + \alpha_{11}\mu + o(\mu), \\ \beta_1(\mu) & = & \beta_1^c + \beta_{11}\mu + o(\mu). \end{array}$$

We denote by $\boldsymbol{t}$ the vector $\boldsymbol{t}= (a_{11}, b_{11}, \alpha_{11}, \beta_{11})$, so that we have $\Lambda(\mu) = \Lambda^c + \mu\boldsymbol{t} + \boldsymbol{o(\mu)}$. We also introduce then the parameter ε by the scaling

$$\varepsilon = \sqrt{|\mu|}, \quad \mu = \pm\varepsilon^2.$$

So unless $\boldsymbol{t}$ is in the tangent space of the boundary at Λ^c, we cross $\mathscr{R}^c$ at $\mu = 0$ with direction $\boldsymbol{t}$. We have

$$\Lambda \in \mathscr{R}^c \Longleftrightarrow G(\Lambda) = 0,$$

where

$$G(\Lambda) = a_1^2\beta_1 + a_1\beta_1^2 - \alpha_1 b_1(a_1 + \beta_1) + a_1. \tag{6.11}$$

Thus $\mathscr{R}^+$ and $\mathscr{R}^-$ can be written as

$$\mathscr{R}^- = \left\{\Lambda \in \mathbb{R}_*^{-4} \ / \ G(\Lambda) < 0\right\}$$
$$\mathscr{R}^+ = \left\{\Lambda \in \mathbb{R}_*^{+4} \ / \ G(\Lambda) > 0\right\}$$

The gradient vector at a point $\Lambda^c \in \mathscr{R}^c$, is given by

$$\nabla G(\Lambda^c) = \begin{pmatrix} 2a_1^c\beta_1^c + \beta_1^{c2} - \alpha_1^c b_1^c + 1 \\ -\alpha_1^c(a_1^c + \beta_1^c) \\ -b_1^c(a_1^c + \beta_1^c) \\ 2\beta_1^c a_1^c + a_1^{c2} - \alpha_1^c b_1^c \end{pmatrix}.$$

So if we want Λ to cross $\mathscr{R}^c$ at $\mu = 0$, we have to take $\boldsymbol{t}$ such that $\boldsymbol{t}.\nabla G(\Lambda^c) \neq 0$. Moreover, we have

$$G(\Lambda) = G(\Lambda^c) + \mu\,\boldsymbol{t}.\nabla G(\Lambda^c) + \boldsymbol{o(\mu)} = \mu\,\boldsymbol{t}.\nabla G(\Lambda^c) + \boldsymbol{o(\mu)}.$$

Thus we conclude that for small μ, $G(\Lambda)$ is of sign $\mu\,\boldsymbol{t}.\nabla G(\Lambda^c)$:

- if $\mu\,\boldsymbol{t}.\nabla G(\Lambda^c) < 0$, then $\Lambda(\mu) \in \mathscr{R}^-$ (stability at the origin),
- if $\mu\,\boldsymbol{t}.\nabla G(\Lambda^c) > 0$, then $\Lambda(\mu) \in \mathscr{R}^+$ (instability at the origin).

We re-formulate the result of the stability analysis in term of μ : depending on the vector $\boldsymbol{t}$ and on the sign of μ, $\Lambda(\mu)$ is in $\mathscr{R}^+$ or $\mathscr{R}^-$ for $\mu \neq 0$ and $\Lambda(0) = \Lambda^c$ is in $\mathscr{R}^c$. And thus at $\mu = 0$ we have a change of stability at the origin (see figure 6.2). Moereover, we have

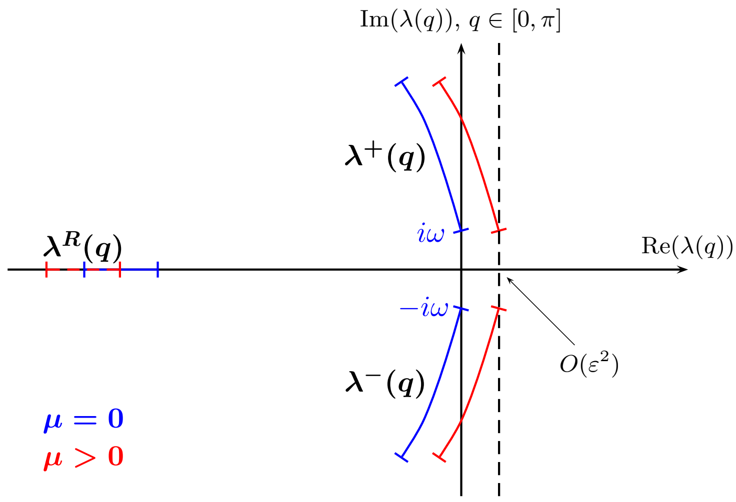


FIGURE 6.2 : Spectrum of $\mathscr{L}$

Theorem 6.6. *For $\mu = 0$, the spectrum of $\mathscr{L}$ has a complex pair of elements crossing the imaginary axis at the points $\pm i\omega$ with*

$$\omega = \sqrt{a_1^c \beta_1^c - \alpha_1^c b_1^c + 1}. \tag{6.12}$$

Proof: We have seen that for $\mu = 0$, Λ lies in the critical variety $\mathscr{R}^c$ and all the roots of the polynomials $P(\lambda, q)$ have nonpositive real part for q in $]0, \pi]$. Let us consider $q = 0$. Then we have a purely imaginary root $\lambda = i\omega$ for $P(\lambda, q)$, if and only if

$$P(i\omega, 0) = 0 \Longleftrightarrow \begin{cases} \omega^2 = a_1^c \beta_1^c - \alpha_1^c b_1^c + 1 \\ \omega^2 = \dfrac{\beta_1^c}{(\beta_1^c + a_1^c)} \end{cases} \Longleftrightarrow \omega = \pm\sqrt{a_1^c \beta_1^c - \alpha_1^c b_1^c + 1},$$

since for Λ^c in $\mathscr{R}^c$, we have $(a_1^c + \beta_1^c)\left(\frac{\alpha_1^c b_1^c}{a_1^c} - \beta_1^c\right) = 1$.

$\square$

Remark 6.7. *We do not have a Hopf bifurcation, because the pair of eigenvalues crossing the imaginary axis is not isolated from the other elements in the spectrum. But for fixed q we have a Hopf bifurcation.*

Shape of the spectrum Figure 6.3 gives the shape of the spectrum for $\ell = 1$, $\Lambda^c = (-1, -1.5, -1, -1)$, $\boldsymbol{t}= (1, 1, 0, 0)$, $\mu = 0$ (black curve) and $\mu < 0$ (red curve), $\mu > 0$ (blue curve). In this case we have $\nabla G(\Lambda^c) = (2.5, -2, -3, 1.5)$, and so $\boldsymbol{t}.\nabla G(\Lambda^c) > 0$, thus for $\mu > 0$ the origin is unstable.

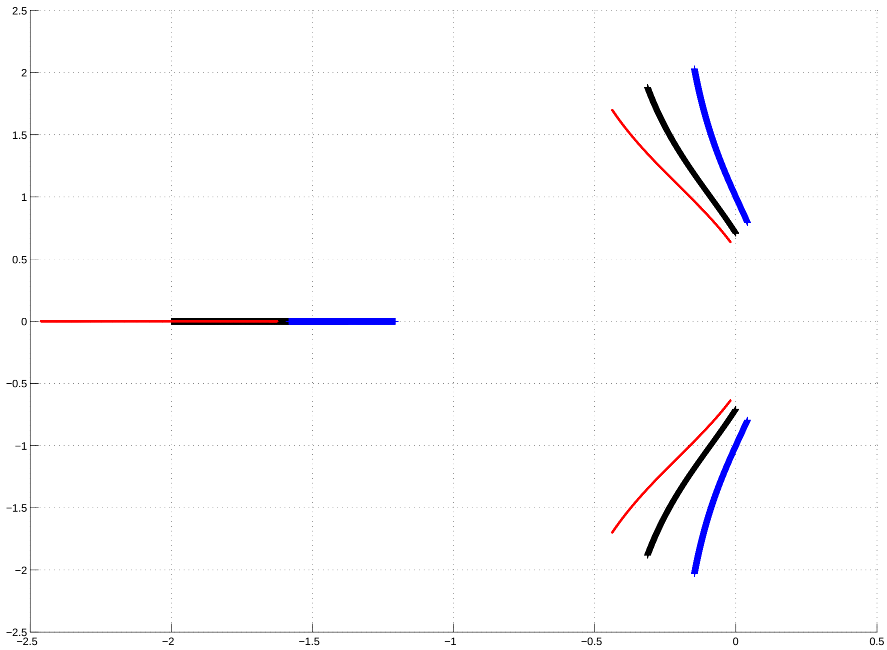


FIGURE 6.3 : Spectrum $\Sigma_{\mathscr{L}}$ for $\mu < 0$, $\mu = 0$ and $\mu > 0$.

Regularity of the roots of $P(X, q)$. To conclude with the spectral analysis, we give some results about regularity of the roots of $P(X, q)$.

Through the parameter Λ, $P(X, q)$ depends also on $\mu = \pm\varepsilon^2$. So from now on, we write $P(X, q, \mu)$ or $P(X, q, \varepsilon^2)$ to point out the dependence of the polynomial P with respect to the continuous parameter q and to the bifurcation parameter $\mu = \pm\varepsilon^2$.
We need regularity to do some estimates of $e^{t\mathscr{L}}$ in Chapter 8. For this, we first investigate the possibility of collision between the roots of $P(X, q, \mu)$, *i.e.* we investigate the possibility of roots of multiplicity 2.

Lemma 6.8. *For fixed q in $[0, \pi]$, $P(X, q)$ admits a double root if and only if Λ belongs to the q-parametrized affine variety $\mathcal{S}_q$ given by the equation*

$$-33\sigma_1^2\sigma_2^2 + 54\sigma_1\sigma_2\sigma_3 + 8\sigma_2\sigma_1^4 - 36\sigma_1^3\sigma_3 + 36\sigma_2^3 + 243\sigma_3^2 = 0,$$

where

$$\begin{aligned}
\sigma_1 &= a_1 + \beta_1, \\
\sigma_2 &= a_1\beta_1 - \alpha_1 b_1 + \Omega_q, \\
\sigma_3 &= \beta_1\Omega_q.
\end{aligned}$$

Proof: The polynomial $P(X,q)$ has a double root if and only if there exists x_0 such that $P(x_0,q) = P'(x_0,q) = 0$. Let us write $P(X) = P'(X)Q(X) + R(X)$ where $R(X)$ is a polynomial of degree less than 1. A simple calculation leads to

$$R(X,q) = \frac{2}{3}(\sigma_2 - \frac{1}{3}\sigma_1^2)X - \sigma_3 + \frac{1}{9}\sigma_1\sigma_2.$$

If such an x_0 exists, then it is a root of R. And so we have

$$x_0 = \frac{1}{2}\frac{\sigma_1\sigma_2 - 9\sigma_3}{3\sigma_2 - \sigma_1^2}. \tag{6.13}$$

Let us then write that x_0 is a root of $P'(X)$:

$$\begin{aligned}
P'(x_0) = 0 \iff& \\
&3(\sigma_1\sigma_2 - 9\sigma_3)^2 - 4\sigma_1(3\sigma_2 - \sigma_1^2)(\sigma_1\sigma_2 - 9\sigma_3) + 4\sigma_2(3\sigma_2 - \sigma_1^2)^2 = 0, \\
&\qquad\iff \\
&-33\sigma_1^2\sigma_2^2 + 54\sigma_1\sigma_2\sigma_3 + 8\sigma_2\sigma_1^4 - 36\sigma_1^3\sigma_3 + 36\sigma_2^3 + 243\sigma_3^2 = 0.
\end{aligned}$$

□

We denote by $\mathcal{S}_0$ this variety for $q = 0$. Then $\mathcal{S}_0$ is not equal to $\mathscr{R}^c$. Indeed, for instance we have $\Lambda^c = (-1, -1.5, -1, -1) \in \mathscr{R}^c$ but $\Lambda^c \notin \mathcal{S}_0$. Thus there exists a neighbourhood of Λ^c in $\mathscr{R}^c$ and of $q = 0$, for which $P(X,q)$ has simple roots.
The result of regularity is then given by the following proposition.

Proposition 6.9

Let consider Λ^c lying in $\mathscr{R}^c$ with $\Lambda^c \notin \mathcal{S}_0$. The roots $\lambda^{\pm,R}(0,\mu)$ of $P(\lambda,0,\mu)$ are smooth functions of μ in a neighbourhood of 0. Moreover, we have

$$Re\,\lambda^+(0,\mu) = O(\mu).$$

Proof: Since there is no collision, $P(\lambda,0,0)$ has three simple roots. Let us denote them $\lambda_0^\pm \in \mathbb{C}$ and $\lambda_0^R \in \mathbb{R}^-$. Then, as $\partial_\lambda P(\lambda_0^{\pm,R},0,0) \neq 0$, we deduce by the Implicit Function Theorem : there exists a neighbourhood $\mathcal{V}^{\pm,R}$ of $(\lambda,\mu) = (\lambda_0^{\pm,R},0)$, such that for all $(\lambda,\mu) \in \mathcal{V}^{\pm,R}$, we have

$$P(\lambda,0,\mu) = 0 \iff \lambda = \lambda^{\pm,R^*}(\mu),$$

where $\lambda^{\pm,R^*}$ is a smooth function (since Λ is a smooth function of μ and thus P also).

Hence can write

$$\lambda^+(0,\mu) \quad = \quad \lambda^+(0,0) + \mu\frac{\partial\lambda^+}{\partial\mu}(0,0) + o(\mu).$$

We calculate $\dfrac{\partial\lambda^+}{\partial\mu}(0,0)$ from the equation

$$P(\lambda^+(0,\mu),0,\mu) = 0 \Longrightarrow \frac{\partial\lambda^+}{\partial\mu}(0,\mu) = -\frac{\frac{\partial P}{\partial\mu}(\lambda^+(0,\mu),0,\mu)}{\frac{\partial P}{\partial\lambda}(\lambda^+(0,\mu),0,\mu)}.$$

Since $\lambda^+(0,0) = i\omega$, it holds

$$\frac{\partial \lambda^+}{\partial \mu}(0,0) = -\frac{\frac{\partial P}{\partial \mu}(i\omega,0,0)}{\frac{\partial P}{\partial \lambda}(i\omega,0,0)}.$$

Moreover we have

$$\begin{aligned}
\frac{\partial P}{\partial \lambda}(i\omega,0,0) &= -3i\omega^2 - 2(a_1^c + \beta_1^c)i\omega + a_1^c\beta_1^c - \alpha_1^c b_1^c + 1,\\
\frac{\partial P}{\partial \mu}(i\omega,0,0) &= \omega^2(a_{11} + \beta_{11}) + i\omega(a_1^c\beta_{11} + a_{11}\beta_1^c - \alpha_{11}b_1^c - \alpha_1^c b_{11}) - \beta_{11}.
\end{aligned}$$

It follows

$$\frac{\partial \lambda^+}{\partial \mu}(0,0) = \frac{\omega^2(a_{11} + \beta_{11}) - \beta_{11} + i\omega\left(a_1^c\beta_{11} + a_{11}\beta_1^c - \alpha_{11}b_1^c - \alpha_1^c b_{11}\right)}{2\omega[\omega + i(a_1^c + \beta_1^c)]}.$$

Thus we deduce

$$\begin{aligned}
\lambda^+(0,\mu) &= i\omega\\
&\quad + \frac{\omega^2(a_{11} + \beta_{11}) - \beta_{11} + i\omega\left(a_1^c\beta_{11} + a_{11}\beta_1^c - \alpha_{11}b_1^c - \alpha_1^c b_{11}\right)}{2\omega[\omega + i(a_1^c + \beta_1^c)]}\mu + o(\mu),
\end{aligned}$$

and so

$$\begin{aligned}
\mathbf{Re}\,\lambda^+(0,\mu) &= \frac{1}{2}\frac{\omega^2(a_{11} + \beta_{11}) - \beta_{11} + (a_1^c + \beta_1^c)(a_1^c\beta_{11} + a_{11}\beta_1^c - \alpha_{11}b_1^c - \alpha_1^c b_{11})}{\omega^2 + (a_1^c + \beta_1^c)^2}\mu\\
&\quad + o(\mu)
\end{aligned}$$

Thus in the spectrum of $\mathscr{L}$, $\lambda^+(0,\mu)$ corresponds to the larger real part, and is of order $O(\mu)$.

□

Remark 6.10. *To prove this result, we only used that $P(X,0,0)$ has no root of multiplicity 2 for some $\Lambda^c \in \mathscr{R}^c$. But in Chapter 8, Section 8.7, we will require no double roots for $P(X,q,\mu)$ for all q in $[0\,,\pi]$ and for μ close to zero (Λ close to Λ^c). Thus we have to make sure that we can find such points Λ^c in $\mathscr{R}^c$. For instance, this is the case with our first example, $\Lambda^c = (-1,-1.5,-1,-1)$, for which this condition is satisfied : $P(X,q,0)$ have 3 simple roots for all $q \in [0\,,\pi]$. And so there is no collision in the spectrum of $\mathscr{L}$ (see figure 6.3).*

Chapitre 7

Formal derivation of the Ginzburg-Landau equation as the amplitude equation

7.1 Formal series ansatz and notations

The purpose of this section is to answer to this question :

What conditions on the amplitude A of the modulated wave (5.5)*, enable us to expect that* (5.5) *is an approximation of the solutions to* (5.3) *in at least a long time interval ?*

Theorem 7.2 answers to this question : a complex Ginzburg-Landau equation appears as a compatibility condition if we insert (5.5) into the equations.

7.1.1 Formal series ansatz

So let us consider a solution close to a modulated wave of the form (5.5). We will now give a form to this solution in order to substitute it into the equation (5.3).

We first notice that in our case, we have $C_{gr} = 0$. Indeed, defined C_{gr} the following way

$$C_{gr} = -i\frac{\partial \lambda}{\partial q}(0,0).$$

We obtain this derivative again with the implicit equation

$$P(\lambda(\mu,q),\mu,q) = 0 \Longrightarrow \frac{\partial \lambda}{\partial q}(0,0) = \frac{\partial_q P(i\omega,0,0)}{\partial_X P(i\omega,0,0)}.$$

Since $i\omega$ is a simple root for $\mu = 0$, $q = 0$, it follows that $\partial_X P(i\omega,0,0) \neq 0$ and an easy calculation leads to $\partial_q P(i\omega,0,0) = 0$. Thus it implies that the group velocity of our modulated wave ansatz is zero.

We thus are looking for an ansatz of type

$$\begin{cases} v_j(t) = \varepsilon A\left(\varepsilon^2 t, \varepsilon j\right) e^{i\omega t} + c.c. + O(\varepsilon^2), \\ \psi_j(t) = \varepsilon B\left(\varepsilon^2 t, \varepsilon j\right) e^{i\omega t} + c.c. + O(\varepsilon^2). \end{cases} \tag{7.1}$$

We notice first that the nonlinear terms in (5.3) will make appear the terms $e^{2i\omega t}$, $e^{3i\omega t}, \ldots$ with amplitude ε^2, $\varepsilon^3 \ldots$ To take this into account we will consider the following formal series as our ansatz

$$\boxed{\begin{cases} v_j(t) = \displaystyle\sum_{k=0}^{+\infty} \sum_{n\in\mathbb{Z}} \varepsilon^k A_{kn}(\tau,\xi) E^n(t), \\ \psi_j(t) = \displaystyle\sum_{k=0}^{+\infty} \sum_{n\in\mathbb{Z}} \varepsilon^k B_{kn}(\tau,\xi) E^n(t), \end{cases}} \tag{7.2}$$

where

$$\boxed{\begin{cases} \tau = \varepsilon^2 t \quad \text{(slow time variable)}, \\ \xi = \varepsilon j \quad \text{(slow space variable)}, \\ E(t) = e^{i\omega t} \end{cases}} \tag{7.3}$$

and where the amplitudes A_{kn}, B_{kn} are complex but satisfying

$$A_{k-n} = \overline{A_{kn}}, \quad B_{k-n} = \overline{B_{kn}}, \quad \forall k \in \mathbb{N},\ n \in \mathbb{Z},$$

since we are looking for real solutions.

Remark 7.1. *The time scale $\tau = \varepsilon^2 t$ is not surprising. Indeed, it comes from the linear problem. Since $\Lambda = \Lambda^c + \rho\varepsilon^2\Lambda_T$, we can write the linear operator $\mathscr{L}$ given by (6.2) as $\mathscr{L} = \mathscr{L}_0 + \varepsilon^2\mathscr{L}_1$. And then it appears a term $e^{c\varepsilon^2 t}$ in the solutions of the linear equation. So $t \sim \frac{1}{\varepsilon^2}$ is the required time scale such that the solutions of the linear problem remain of order $O(1)$.*
For the space scale, we refer to the Fourier analysis lead in [Sch01].

The calculation is the object of the following subsection.

7.2 Formal derivation

To fix the ideas, we take a parameter $\Lambda^c \in \mathscr{R}^c$ and we choose a vector $\boldsymbol{t} = (a_{11}, b_{11}, \alpha_{11}, \beta_{11})$ such that $\boldsymbol{t}.\nabla G(\Lambda^c) > 0$ (G is given by (6.11)). So the origin is unstable for $\mu > 0$. Consequently, from now on, we consider $\mu = +\varepsilon^2$, so that we are on the unstable side. And we are going to substitute the ansatz (7.2) into the system (5.3), which we recall here

$$\begin{cases} \ddot{v}_j + v_j = (\Delta_d\, v)_j + (a_1^c + a_{11}\varepsilon^2 + o(\varepsilon^2))\dot{v}_j + a_2\dot{v}_j^2 + a_3\dot{v}_j^3 + N_1(\dot{v}_j) \\ \qquad\qquad + (b_1^c + b_{11}\varepsilon^2 + o(\varepsilon^2))\psi_j + b_2\psi_j^2 + b_3\psi_j^3 + N_2(\psi_j), \\ \dot{\psi}_j = (\alpha_1^c + \alpha_{11}\varepsilon^2 + o(\varepsilon^2))\dot{v}_j + (\beta_1^c + \beta_{11}\varepsilon^2 + o(\varepsilon^2))\psi_j + \gamma_{11}\psi_j\dot{v}_j, \end{cases}$$

for j in $\mathbb{Z}$ and where $N_1(x), N_2(x) = O(x^4)$. We also recall the formula (6.12)

$$\omega = \sqrt{a_1^c\beta_1^c - \alpha_1^c b_1^c + 1}.$$

We now state the main result of this section, which gives the amplitude equation that is supposed to describe the dynamics of small solutions.

Theorem 7.2

We denote by $E(t) = e^{i\omega t}$. Suppose that (5.3) has solutions of the form

$$\begin{cases} v_j(t) = v_j^A(t) + O(\varepsilon^2), \\ \psi_j(t) = \psi_j^B(t) + O(\varepsilon^2), \end{cases} \quad \textit{with} \quad \begin{cases} v_j^A(t) = \varepsilon A(\tau,\xi)E(t) + c.c., \\ \psi_j^B(t) = \varepsilon B(\tau,\xi)E(t) + c.c., \end{cases}$$

for all $j \in \mathbb{Z}$ and all $\varepsilon \in (0, \varepsilon_0)$, where A is a smooth fonction $[0,T] \times \mathbb{R} \longrightarrow \mathbb{C}$. Then A is solution to the complex scalar Ginzburg-Landau equation

$$\boxed{\partial_\tau A = (c+id)A + (\nu + i\alpha)\partial_{\xi\xi}A + (a+ib)|A|^2 A,}$$

where the coefficients are given by formulas (7.12)-(7.13) and B is given by

$$B = \frac{i\omega\alpha_1^c}{i\omega - \beta_1^c}A.$$

Proof: We will prove that given A solution to the Ginzburg-Landau equation, we can determine all the other amplitudes A_{kn} for $1 \leq k \leq 3$ and $-k \leq n \leq k$. Let us consider v and ψ given by the formal series (7.2), and let us calculate each term at order ε^3. We obtain

$$\begin{aligned}
\dot v_j(t) &= i\omega A_{11}\varepsilon E + i\omega A_{21}\varepsilon^2 E + 2i\omega A_{22}\varepsilon^2 E^2 + \partial_\tau A_{10}\varepsilon^3 E^0 \\
&\quad +(\partial_\tau A_{11} + 3i\omega A_{31})\varepsilon^3 E + 2i\omega A_{32}\varepsilon^3 E^2 + 3i\omega A_{33}\varepsilon^3 E^3 \\
&\quad + c.c. + o(\varepsilon^3), \\
\dot v_j^2(t) &= 2\omega^2 |A_{11}|^2 \varepsilon^2 E^0 - \omega^2 A_{11}^2 \varepsilon^2 E^2 + 2\omega^2(\overline{A_{11}}A_{21} + A_{11}\overline{A_{21}})\varepsilon^3 E^0 \\
&\quad +4\omega^2 \overline{A_{11}} A_{22}\varepsilon^3 E - 2\omega^2 A_{11}A_{21}\varepsilon^3 E^2 - 4\omega^2 A_{11}A_{22}\varepsilon^3 E^3 \\
&\quad + c.c. + o(\varepsilon^3), \\
\dot v_j^3(t) &= 3i\omega^3 |A_{11}|^2 A_{11}\varepsilon^3 E - i\omega^3 A_{11}^3 \varepsilon^3 E^3 + c.c. + o(\varepsilon^3), \\
\ddot v_j(t) &= -\omega^2 A_{11}\varepsilon E - \omega^2 A_{21}\varepsilon^2 E - 4\omega^2 A_{22}\varepsilon^2 E^2 \\
&\quad +(2i\omega\partial_\tau A_{11} - \omega^2 A_{31})\varepsilon^3 E - 4\omega^2 A_{32}\varepsilon^3 E^2 - 9\omega^2 A_{33}\varepsilon^3 E^3 \\
&\quad + c.c. + o(\varepsilon^3), \\
\Delta_d\, v_j(t) &= \partial_{\xi\xi} A_{10}\varepsilon^3 E^0 + \partial_{\xi\xi} A_{11}\varepsilon^3 E + c.c. + o(\varepsilon^3),
\end{aligned}$$

$$\begin{aligned}
\psi_j^2(t) &= (B_{10}^2 + 2|B_{11}|^2)\varepsilon^2 E^0 + 2B_{10}B_{11}\varepsilon^2 E + B_{11}^2\varepsilon^2 E^2 \\
&\quad +2(B_{10}B_{20} + B_{11}\overline{B_{21}} + \overline{B_{11}}B_{21})\varepsilon^3 E^0 \\
&\quad +2(B_{10}B_{21} + B_{10}B_{20} + \overline{B_{11}}B_{22})\varepsilon^3 E \\
&\quad +2(B_{10}B_{22} + B_{11}B_{21})\varepsilon^3 E^2 + 2B_{11}B_{22}\varepsilon^3 E^3 + .c.c. + o(\varepsilon^3), \\
\psi_j^3(t) &= B_{10}^3\varepsilon^3 E^0 + 3(B_{10}^2 B_{11} + B_{11}^2\overline{B_{11}})\varepsilon^3 E + 3B_{11}^2 B_{10}\varepsilon^3 E^2 \\
&\quad +B_{11}^3\varepsilon^3 E^3 + c.c. + o(\varepsilon^3), \\
\psi_j(t)\dot v_j(t) &= i\omega(A_{11}\overline{B_{11}} - B_{11}\overline{A_{11}})\varepsilon^2 E^0 + i\omega B_{10}A_{11}\varepsilon^2 E + i\omega B_{11}A_{11}\varepsilon^2 E^2 \\
&\quad +iw\left(\overline{B_{11}}A_{21} - B_{11}\overline{A_{21}} + A_{11}\overline{B_{21}} - \overline{A_{11}}B_{21}\right)\varepsilon^3 E^0 \\
&\quad +i\omega\left(B_{10}A_{21} + 2A_{22}\overline{B_{11}} + A_{11}B_{20} - \overline{A_{11}}B_{22}\right)\varepsilon^3 E^1 \\
&\quad +i\omega\left(2B_{10}A_{22} + B_{11}A_{21} + A_{11}B_{21}\right)\varepsilon^3 E^2 \\
&\quad +i\omega\left(2A_{22}B_{11} + A_{11}B_{22}\right)\varepsilon^3 E^3 + c.c. + o(\varepsilon^3).
\end{aligned}$$

We now substitute this into equations (5.3) and equate each order $\varepsilon^k E^n$, for $1 \le k \le 3$ and $-k \le n \le k$. It gives the following results.

(i) $\boldsymbol{\varepsilon E^0}$:

$$\begin{cases} A_{10} = b_1^c B_{10}, \\ \quad 0 = \beta_1^c B_{10}. \end{cases}$$

Thus we deduce

$$\boxed{A_{10} = B_{10} = 0,} \tag{7.4}$$

which simplifies the next orders.

(ii) $\varepsilon \boldsymbol{E}$ **:**

$$\begin{cases} -\omega^2 A_{11} + A_{11} = a_1^c i\omega A_{11} + b_1^c B_{11}, \\ \qquad\qquad i\omega B_{11} = \alpha_1^c i\omega A_{11} + \beta_1^c B_{11}. \end{cases}$$

We deduce from the second equation that

$$\boxed{B_{11} = \Omega_{11} A_{11}, \quad \Omega_{11} = \frac{i\omega\alpha_1^c}{i\omega - \beta_1^c} \in \mathbb{C}.} \tag{7.5}$$

Then the first equation can be written as

$$P(i\omega, 0, 0) A_{11} = 0,$$

which is satisfied for any A_{11} since $P(i\omega, 0, 0) = 0$.

(iii) $\varepsilon^2 \boldsymbol{E}^0$ **:**

$$\begin{cases} A_{20} = b_1^c B_{20} + 2a_2\omega^2 |A_{11}|^2 + 2b_2 |B_{11}|^2, \\ \quad 0 = \beta_1^c B_{20} + \gamma_{11} i\omega (A_{11}\overline{B_{11}} - B_{11}\overline{A_{11}}). \end{cases}$$

Given (7.5), we obtain here

$$\boxed{\begin{cases} B_{20} = \Omega_{20} |A_{11}|^2, \quad \Omega_{20} = \dfrac{2\omega\gamma_{11}}{\beta_1^c} \mathrm{Im}\, \Omega_{11} \in \mathbb{R}, \\ A_{20} = \Omega'_{20} |A_{11}|^2, \quad \Omega'_{20} = b_1^c \Omega_{20} + 2a_2\omega^2 + 2b_2 |\Omega_{11}|^2 \in \mathbb{R}. \end{cases}} \tag{7.6}$$

(iv) $\varepsilon^2 \boldsymbol{E}^1$ **:**

$$\begin{cases} -\omega^2 A_{21} + A_{21} = i\omega a_1^c A_{21} + b_1^c B_{21}, \\ \qquad\qquad i\omega B_{21} = i\omega \alpha_1^c A_{21} + \beta_1^c B_{21}. \end{cases}$$

The second equation gives

$$\boxed{B_{21} = \Omega_{11} A_{21},} \tag{7.7}$$

and it follows that the first equation can we be written as

$$P(i\omega, 0, 0) A_{21} = 0,$$

which is satisfied for any A_{21}.

(v) $\varepsilon^2 \boldsymbol{E}^2$ **:**

$$\begin{cases} -4\omega^2 A_{22} + A_{22} = 2i\omega a_1^c A_{22} + b_1^c B_{22} - \omega^2 a_2 A_{11}^2 + b_2 B_{11}^2, \\ \qquad\qquad 2i\omega B_{22} = 2i\omega \alpha_1^c A_{22} + \beta_1^c B_{22} + i\omega\gamma_{11} B_{11} A_{11}. \end{cases}$$

We obtain

$$\boxed{\begin{cases} A_{22} = \Omega'_{22} A_{11}^2, \quad \Omega'_{22} = \dfrac{Q(i\omega)}{P(2i\omega,0,0)} \in \mathbb{C}, \\ B_{22} = \Omega_{22} A_{11}^2, \quad \Omega_{22} = \dfrac{2i\omega\Omega'_{22}\alpha_1^c + i\omega\gamma_{11}\Omega_{11}}{2i\omega - \beta_1^c} \in \mathbb{C}, \end{cases}} \tag{7.8}$$

with

$$Q(i\omega) = -2i\omega^3 a_2 + a_2\beta_1^c\omega^2 + i\omega\Omega_{11}(\gamma_{11}b_1^c + 2b_2\Omega_{11}) - \beta_1^c b_2 \Omega_{11}^2.$$

(vi) $\boldsymbol{\varepsilon^3 E^0}$ **:**

$$\begin{cases} A_{30} = b_1^c B_{30} + 2a_2\omega^2(\overline{A_{11}}A_{21} + A_{11}\overline{A_{21}}) + 2b_2(B_{11}\overline{B_{21}} + \overline{B_{11}}B_{21}), \\ \quad 0 = \beta_1^c B_{30} + i\omega\gamma_{11}(\overline{B_{11}}A_{21} - B_{11}\overline{A_{21}} + A_{11}\overline{B_{21}} - \overline{A_{11}}B_{21}). \end{cases}$$

We obtain

$$\boxed{\begin{cases} B_{30} = 4\Omega_{30}\mathrm{Re}\,(A_{11}\overline{A_{21}}), \quad \Omega_{30} = \dfrac{\gamma_{11}\omega}{\beta_1^c}\mathrm{Im}\,\overline{\Omega_{11}} \in \mathbb{R}, \\ A_{30} = 4\Omega'_{30}\mathrm{Re}\,(A_{11}\overline{A_{21}}), \quad \Omega'_{30} = \Omega_{30} + a_2\omega^2 + b_2|\Omega_{11}|^2 \in \mathbb{R}. \end{cases}} \tag{7.9}$$

(vii) $\boldsymbol{\varepsilon^3 E^1}$ **:**

$$\begin{cases} -\omega^2 A_{31} + 2i\omega\partial_\tau A_{11} + A_{31} = \ell^2\partial_{\xi\xi}A_{11} + a_1^c(\partial_\tau A_{11} + i\omega A_{31}) \\ \qquad + b_1^c B_{31} + 4a_2\omega^2\overline{A_{11}}A_{22} + 2b_2\overline{B_{11}}B_{22} \\ \qquad + 3b_3 B_{11}^2\overline{B_{11}} + 3i\omega^3 a_3 A_{11}|A_{11}|^2 \\ \qquad + b_{11}\Omega_{11}A_{11} + i\omega a_{11}A_{11}, \\ i\omega B_{31} + \partial_\tau B_{11} = \alpha_1^c(\partial_\tau A_{11} + i\omega A_{31}) + \beta_1^c B_{31} \\ \qquad + i\omega\gamma_{11}(2A_{22}\overline{B_{11}} + A_{11}B_{20} - \overline{A_{11}}B_{22}) \\ \qquad + \beta_{11}B_{11} + i\omega\alpha_{11}A_{11}. \end{cases}$$

Given formulas (7.5),(7.8) and $P(i\omega,0,0) = 0$, we first deduce from the second equation that

$$\boxed{B_{31} = \lambda_1 A_{31} + \lambda_2\partial_\tau A_{11} + \lambda_3 A_{11}|A_{11}|^2 + \lambda_4 A_{11},} \tag{7.10}$$

with

$$\begin{aligned} \lambda_1 &= \frac{i\omega\alpha_1^c}{i\omega - \beta_1^c}, \\ \lambda_2 &= \frac{\alpha_1^c - \Omega_{11}}{i\omega - \beta_1^c}, \\ \lambda_3 &= \frac{i\omega\gamma_{11}(2\Omega_{22}\overline{\Omega_{11}} + \Omega_{20} - \Omega'_{22})}{i\omega - \beta_1^c}, \\ \lambda_4 &= \frac{\beta_{11}\Omega_{11} + i\omega\alpha_{11}}{i\omega - \beta_1^c}. \end{aligned}$$

Substituting this into the first equation gives the following PDE

$$\boxed{\partial_\tau A_{11} = (c+id)A_{11} + (\nu + i\alpha)\partial_{\xi\xi}A_{11} + (a+ib)|A_{11}|^2 A_{11},} \tag{7.11}$$

with

$$\begin{aligned} c+id &= \frac{(i\omega - \beta_1^c)b_{11}\Omega_{11} + b_1^c(\beta_{11}\Omega_{11} + \alpha_{11}i\omega)}{(2i\omega - a_1^c)(i\omega - \beta_1^c) - b_1^c(\alpha_1^c - \Omega_{11})}, \\ \nu + i\alpha &= \frac{(i\omega - \beta_1^c)\ell^2}{(2i\omega - a_1^c)(i\omega - \beta_1^c) - b_1^c(\alpha_1^c - \Omega_{11})}, \end{aligned} \tag{7.12}$$

and

$$\begin{aligned} a+ib &= \frac{(i\omega - \beta_1^c)(4a_2\omega^2\Omega_{22} + 2b_2\Omega'_{22}\overline{\Omega_{11}} + 3b_3\Omega_{11}|\Omega_{11}|^2 + 3i\omega^3 a_3)}{(a_1^c - 2i\omega)(i\omega - \beta_1^c) + b_1^c(\alpha_1^c - \Omega_{11})} \\ &\quad + \frac{i\omega b_1^c\gamma_{11}(2\Omega_{22}\overline{\Omega_{11}} + \Omega_{20} - \Omega'_{22})}{(a_1^c - 2i\omega)(i\omega - \beta_1^c) + b_1^c(\alpha_1^c - \Omega_{11})}. \end{aligned} \tag{7.13}$$

(vi) $\boldsymbol{\varepsilon^3 E^2}$ **:**

$$\begin{cases} -4\omega^2 A_{32} + A_{32} = 2i\omega a_1^c A_{32} + b_1^c B_{32} - 2\omega^2 a_2 A_{11}A_{21} + 2b_2 B_{11}B_{21}, \\ \qquad\qquad 2i\omega B_{32} = 2i\omega\alpha_1^c A_{32} + \beta_1^c B_{32} + i\omega\gamma_{11}(B_{11}A_{21} + A_{11}B_{21}). \end{cases}$$

It gives

$$\boxed{\begin{cases} A_{32} = \Omega'_{32}A_{11}A_{21}, \quad \Omega'_{32} = \dfrac{2(b_2\Omega_{11}^2 - a_2\omega^2)(2i\omega - \beta_1^c) + 2i\omega b_1^c\gamma_{11}\Omega_{11}}{(1 - 4\omega^2 - 2i\omega a_1^c)(2i\omega - \beta_1^c) - i\omega b_1^c\alpha_1^c} \in \mathbb{C}, \\ B_{32} = \Omega_{32}A_{11}A_{21}, \quad \Omega_{32} = \dfrac{2i\omega}{2i\omega - \beta_1^c}(\alpha_1^c\Omega'_{32} + \gamma_{11}\Omega_{11}) \in \mathbb{C}. \end{cases}} \tag{7.14}$$

(vii) $\boldsymbol{\varepsilon^3 E^3}$ **:**

$$\begin{cases} -9\omega^2 A_{33} + A_{33} = 3i\omega a_1^c A_{33} + b_1^c B_{33} - 4a_2\omega^2 A_{11}A_{22} + 2b_2 B_{11}B_{22} \\ \qquad\qquad - i\omega^3 a_3 A_{11}^3 + b_3 B_{11}^3, \\ \qquad 3i\omega B_{33} = 3i\omega\alpha_1^c A_{33} + \beta_1^c B_{33} + i\omega\gamma_{11}(2A_{22}B_{11} + A_{11}B_{22}). \end{cases}$$

We obtain with (7.8) and (7.5)

$$\boxed{\begin{cases} A_{33} = \Omega'_{33}A_{11}^3, \\ B_{33} = \Omega_{33}A_{11}^3, \end{cases}} \tag{7.15}$$

with

$$\begin{aligned} \Omega'_{33} &= \frac{i\omega b_1^c\gamma_{11}(2\Omega'_{22}\Omega_{11} + \Omega_{22})}{(3i\omega - \beta_1^c)(-9\omega^2 + 1 - 3i\omega a_1^c) - 3i\omega\alpha_1^c b_1^c} \\ &\quad + \frac{(3i\omega - \beta_1^c)(2b_2\Omega_{11}\Omega_{22} - i\omega^3 a_3 + b_3\Omega_{11}^3 - 4a_2\omega^2\Omega'_{22})}{(3i\omega - \beta_1^c)(-9\omega^2 + 1 - 3i\omega a_1^c) - 3i\omega\alpha_1^c b_1^c} \in \mathbb{C}, \\ \Omega_{33} &= \frac{3i\omega\alpha_1^c\Omega'_{33} + i\omega\gamma_{11}(2\Omega'_{22}\Omega_{11} + \Omega_{22})}{3i\omega - \beta_1^c} \in \mathbb{C}. \end{aligned}$$

To sum up, by equating to the order ε^3, we obtained

$$\begin{array}{lll} A_{10} = B_{10} = 0, & B_{11} = \Omega_{11} A_{11}, & B_{20} = \Omega_{20}|A_{11}|^2, \\ A_{20} = \Omega'_{20}|A_{11}|^2, & A_{22} = \Omega'_{22} A_{11}^2, & B_{22} = \Omega_{22} A_{11}^2, \\ A_{33} = \Omega'_{33} A_{11}^3, & B_{33} = \Omega_{33} A_{11}^3, & \end{array} \tag{7.16}$$

B_{31} is a function of A_{11} and $\partial_\tau A_{11}$, and A_{11} satisfies the PDE (7.11) :

$$\partial_\tau A_{11} = (c + id)u + (\nu + i\alpha)\partial_{\xi\xi} A_{11} + (a + ib)|A_{11}|^2 A_{11}.$$

We have no condition at this order on A_{21} and A_{31}. Lastly, B_{21}, B_{30}, A_{30}, A_{32}, B_{32} are functions of A_{21} and A_{11}

$$\begin{array}{lll} B_{21} = \Omega_{11} A_{21}, & B_{30} = 4\Omega_{30} \mathrm{Re}\,(A_{11}\overline{A_{21}}), & A_{30} = 4\Omega'_{30} \mathrm{Re}\,(A_{11}\overline{A_{21}}), \\ A_{32} = \Omega'_{32} A_{11} A_{21}, & B_{32} = \Omega_{32} A_{11} A_{21}. & \end{array} \tag{7.17}$$

Since, A_{21} and A_{31} are free, we set $\boldsymbol{A_{31} = A_{21} = 0}$. It follows from (7.16) and (7.17) that

$$\left\{ \begin{array}{l} A_{10} = B_{10} = A_{21} = B_{21} = A_{30} = B_{30} = A_{31} = A_{32} = B_{32} = 0, \\ A_{11} \text{ is solution to (7.11)}, \\ B_{11} = \Omega_{11} A_{11}, \;\; B_{20} = \Omega_{20}|A_{11}|^2, \;\; A_{20} = \Omega'_{20}|A_{11}|^2, \;\; A_{22} = \Omega'_{22} A_{11}^2, \\ B_{22} = \Omega_{22} A_{11}^2, \;\; A_{33} = \Omega'_{33} A_{11}^3, \;\; B_{33} = \Omega_{33} A_{11}^3, \\ B_{31} \text{ is a function of } A_{11} \text{ and } \partial_\tau A_{11}. \end{array} \right. \tag{7.18}$$

Consequently, given A_{11} solution to equation (7.11), we can determine all the A_{kn} and B_{kn} for $k \leq 3$, $|n| \leq k$. And the ansatz (7.2) takes the form

$$\boxed{\left\{ \begin{array}{l} v_j^A(t) = \varepsilon A_{11} E + \varepsilon^2 \Omega'_{20}|A_{11}|^2 + \varepsilon^2 \Omega'_{22} A_{11}^2 E^2 + \varepsilon^3 \Omega'_{33} A_{11}^3 E^3 + c.c. + o(\varepsilon^3) \\ \psi_j^B(t) = \varepsilon \Omega_{11} A_{11} E + \varepsilon^2 \Omega_{20}|A_{11}|^2 + \varepsilon^2 \Omega_{22} A_{11}^2 E^2 + \varepsilon^3 B_{31} E \\ \qquad\qquad + \varepsilon^3 \Omega_{33} A_{11}^3 E^3 + c.c. + o(\varepsilon^3). \end{array} \right.} \tag{7.19}$$

□

Chapitre 8

Justification in the cubic case

8.1 Introduction and statement of the main result

8.1.1 The cubic case

The purpose of this section is to prove the validity of the derived amplitude equation. In other words, we want to answer to the following question.

Question : do we have solutions to (5.3) *that are close to the formal approximation* (5.5)*, induced by an amplitude A, solution to the Ginzburg-Landau equation* (7.11) *?*

Indeed, the fact that we can derive formally an amplitude equation does not ensure that there exist solutions of that form. For a counter example of this fact, see [Sch95], where it is shown that the solutions of the original Bénard's problem can behave in some situations in a completely different manner than predicted by the formally derived amplitude Newell-Whitehead equation.
As announced, we restrict here to the cubic case. It means that the nonlinearity has no quadratic term. In other words, we have

$$\boxed{a_2 = b_2 = \gamma_{11} = 0.} \tag{8.1}$$

Under this hypothesis, we get

$$\Omega_{20} = \Omega'_{20} = \Omega_{22} = \Omega'_{22} = \Omega'_{30} = \Omega'_{32} = 0,$$

and the other coefficients simplify as follow

$$\begin{cases} B_{11} = \Omega_{11} A_{11}, \\ B_{31} = \Omega_{31} \partial_\tau A_{11}, \\ A_{33} = \Omega'_{33} A_{11}^3, \\ B_{33} = \Omega_{33} A_{11}^3, \end{cases} \quad \text{with} \quad \begin{cases} \Omega_{11} = \dfrac{i\omega\alpha_1^c}{i\omega - \beta_1^c}, \\ \Omega_{31} = \dfrac{\alpha_1^c - \Omega_{11}}{i\omega - \beta_1^c}, \\ \Omega'_{33} = \dfrac{b_3 \Omega_{11}^3 - i\omega^3 a_3}{1 - 9\omega^2 - 3i\omega a_1^c - \frac{3\alpha_1^c i\omega}{3i\omega - \beta_1^c}}, \\ \Omega_{33} = \dfrac{3i\omega\alpha_1^c \Omega'_{33}}{3i\omega - \beta_1^c}. \end{cases} \tag{8.2}$$

Then the formal approximation at order 3 given by (7.19) simplifies as

$$\boxed{\begin{cases} v_j^A(t) = \varepsilon A(\tau,\xi)E(t) + \varepsilon^3\Omega'_{33}A^3(\tau,\xi)E^3(t) + c.c. + o(\varepsilon^3) \\ \psi_j^B(t) = \varepsilon\Omega_{11}A(\tau,\xi)E(t) + \varepsilon^3\Omega_{31}\partial_\tau A(\tau,\xi)E(t) \\ \qquad\qquad + \varepsilon^3\Omega_{33}A^3(\tau,\xi)E^3(t) + c.c. + o(\varepsilon^3). \end{cases}} \tag{8.3}$$

The complex Ginzburg-Landau amplitude equation of the cubic case is then

$$\boxed{\partial_\tau A = (c+id)A + (\nu + i\alpha)\partial_{\xi\xi}A + (a+ib)|A|^2A,} \tag{8.4}$$

with

$$\begin{aligned} c + id &= \frac{b_{11}\Omega_{11} + b_1^c\frac{\beta_{11}\Omega_{11}+\alpha_{11}i\omega}{i\omega-\beta_1^c}}{2i\omega - a_1^c - b_1^c\Omega_{31}}, \\ \nu + i\alpha &= \frac{\ell^2}{2i\omega - a_1^c - b_1^c\Omega_{31}}, \\ a + ib &= 3\frac{i\omega^3 a_3 + b_3\Omega_{11}|\Omega_{11}|^2}{a_1^c - 2i\omega + b_1^c\Omega_{31}}. \end{aligned} \tag{8.5}$$

8.1.2 Statement of the validity Theorem 8.1

Assumptions and hypotheses We consider the first order differential equation (6.1),

$$\frac{dV}{dt} = \mathscr{L}V + \mathscr{N}(V),$$

in the Banach space $\mathcal{Y} = \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ where $\mathscr{L}$ and $\mathscr{N}$ are given by formulas (6.2).

- Because of the cubic hypothesis, the nonlinearity $\mathscr{N}$ reads

$$\mathscr{N}(V) = \begin{pmatrix} 0 \\ N(V) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ N_1(\dot{v}) + N_2(\psi) \\ 0 \end{pmatrix},$$

  with $N_1(\dot{v}) = a_3\dot{v}^3 + \widehat{N_1}(\dot{v})$, $N_2(\psi) = b_3\psi^3 + \widehat{N_2}(\psi)$ smooth functions in $C^5(\mathbb{R},\mathbb{R})$ satisfying $\widehat{N_1}(\dot{v}) = O(\dot{v}^4)$, $\widehat{N_2}(\psi) = O(\psi^4)$.

- We suppose that the coefficients a_1, b_1, α_1, β_1, a_3, b_3 are real parameters satisfying hypothesis (5.4) and that we are at the threshold of stability :

$$\Lambda = \Lambda^c + {}^T(a_{11}, b_{11}, \alpha_{11}, \beta_{11})\mu, \quad \mu = \pm\varepsilon$$

  for a point Λ^c in the boundary $\mathscr{R}^{c+}$ such that we have no roots of multiplicity 2 for $P(\lambda, q)$ and for all q in $[0\,,\pi]$.

Thus the result of proposition 6.9 holds and we have the following theorem.

Theorem 8.1

We suppose that the previous hypotheses and assumptions are satisfied. Let then consider $\tau_0 > 0$ *and* $A(\tau,\xi) : [0,\tau_0] \times \mathbb{R} \longrightarrow \mathbb{C}$*, a solution of the Ginzburg-Landau amplitude equation given by Theorem 7.2 with initial data* $A(0,.)$ *in* $H^5_\xi(\mathbb{R})$*.*
Let us also define V_A *the formal approximation at order one, induced by* A *:*

$$V_A = \begin{pmatrix} v_A \\ \dot{v}_A \\ \psi_A \end{pmatrix} \quad \textit{with} \quad \begin{cases} v_j^A(t) = \varepsilon A(\tau,\xi)E(t) + c.c., \\ \psi_j^B(t) = \varepsilon \Omega_{11} A(\tau,\xi)E(t) + c.c., \end{cases} \tag{8.6}$$

where $E(t) = e^{i\omega t}$ *and* ω *such that* $i\omega = \lambda^+(0,0)$*.*

Then for all $d_0 > 0$*, there exists* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$*, the following holds : for any solution* V *of* (6.1) *satisfying*

$$\|V(0) - V_A(0)\|_{\mathcal{Y}} \leq d_0 \varepsilon^{\frac{3}{2}}, \tag{8.7}$$

we have

$$\|V(t) - V_A(t)\|_{\mathcal{Y}} \leq c\varepsilon^{\frac{3}{2}}, \quad \textit{for all} \;\; t \in [0\,, \frac{\tau_0}{\varepsilon^2}]. \tag{8.8}$$

This theorem first deserves some remarks and comments.

(i) This theorem says : *if we are near the threshold of instability,* $\Lambda \approx \Lambda^c$*, then we can construct many solutions that are close to a modulated waves family.* This family is the set of waves of the form (8.6), for which the amplitude is a solution of the complex Ginzburg-Landau amplitude equation. In other words, to each solution A of the Ginzburg-Landau equation, corresponds a function V_A, which is a nearby exact solution to the initial problem with a relatively small error.

(ii) These modulated waves have small amplitude of order $O(\varepsilon)$. And then "*close to*" means that the error estimate is less than ε. We obtain in our case an estimate in $\varepsilon^{\frac{3}{2}}$. But to begin, we suppose an error in ε^α. The proof of this theorem will lead us to $\alpha = \frac{3}{2}$.
Inequality (8.8) also says that the error does not amplify in a time-scale of $\frac{1}{\varepsilon^2}$.

(iii) This approximation is valid in a time interval of size $O(\frac{1}{\varepsilon^2})$. Although it is an approximation in a finite time interval, this time interval is very large.

(iv) As explained in the introduction, it is also possible then to construct pseudo-orbits with these solutions, which are valid for infinite time intervals.

(v) The H^5_ξ regularity needed for the initial data comes from the fact that we need in the proof bounds for the L^2_ξ-norm of $\partial^4_\xi A$ and $\partial_{\tau\tau} A$. If $A(0,.) \in H^5_\xi$, then Theorem 8.2 ensures that we have those bounds.

(vi) The error estimate in $O(\varepsilon^{\frac{3}{2}})$ is not optimal. Indeed, we estimate the ℓ^2-norm of sequences of type $A(\tau, \varepsilon j)$, which implies a loss of order $O(\varepsilon^{\frac{1}{2}})$ (see Lemma 8.18). We could obtain a better error estimate in considering ℓ^∞-norms.

8.1.3 Notations

We are going to introduce here some notations, that will be needed throughout the proof of Theorem 8.1.

- $\boldsymbol{V_A}$, given by (8.6) stands for the **first approximation** or **approximation at order 1**, that is, we neglect the $O(\varepsilon^2)$-term.

- To write the order 3-approximation, we introduce $\boldsymbol{W_A}$, which corresponds to **the order 3-terms** in the ansatz (7.19) of $v_j^A(t)$ and $\psi_j^A(t)$:

$$W_A = \begin{pmatrix} w_A \\ \dot{w}_A \\ \phi_A \end{pmatrix},$$

where

$$\begin{cases} w_A^j = \varepsilon^3 \Omega'_{33} A^3(\tau, \xi) E^3(t) + c.c., \\ \phi_A^j(t) = \varepsilon^3 \Omega_{31} \partial_\tau A(\tau, \xi) E(t) + \varepsilon^3 \Omega_{33} A^3(\tau, \xi) E^3(t) + c.c., \end{cases} \tag{8.9}$$

with Ω'_{33}, Ω_{33} and Ω_{31} given by (8.2).

- We denote by $\widetilde{\boldsymbol{V_A}}$, the **approximation at order 3**

$$\widetilde{V_A} = V_A + W_A = \begin{pmatrix} \widetilde{v_A} \\ \widetilde{\dot{v}_A} \\ \widetilde{\psi_A} \end{pmatrix},$$

where

$$\begin{cases} \widetilde{v_{A,j}}(t) = \varepsilon A(\tau, \varepsilon j) E(t) + \varepsilon^3 \Omega'_{33} A^3(\tau, \xi) E^3(t) + c.c., \\ \widetilde{\dot{v}_{A,j}}(t) = i\omega\varepsilon A(\tau, \varepsilon j) E(t) + \varepsilon^3 \partial_\tau A(\tau, \varepsilon j) E(t) + 3i\omega \Omega'_{33} \varepsilon^3 A^3(\tau, \varepsilon j) E^3(t) \\ \qquad\qquad + 3\Omega'_{33} \varepsilon^5 \partial_\tau A(\tau, \varepsilon j) A^2(\tau, \varepsilon j) E^3(t) + c.c, \\ \widetilde{\psi_{A,j}}(t) = \varepsilon \Omega_{11} A(\tau, \varepsilon j) E(t) + \varepsilon^3 \Omega_{31} \partial_\tau A(\tau, \xi) E(t) + \varepsilon^3 \Omega_{33} A^3(\tau, \xi) E^3(t) \\ \qquad\qquad + c.c., \end{cases} \tag{8.10}$$

Thus $\widetilde{V_A}$ is obtained when we neglect the $O(\varepsilon^4)$-terms.

8.2 Strategy of proof of Theorem 8.1

We want to estimate the error $\|V(t) - V_A(t)\|_{\mathcal{Y}}$ between the formal approximation at order one, V_A, and a solution V of (5.3), when the initial condition $V(0)$ is close to $V_A(0)$.

The cornerstone of the proof is to take into account the terms of order 3. In other words, we will introduce the order 3-terms, namely W_A. To estimate the error $\|V(t) - V_A(t)\|_{\mathcal{Y}}$, we thus write

$$\begin{aligned} \|V(t) - V_A(t)\|_{\mathcal{Y}} &= \|V(t) - V_A(t) - W_A(t) + W_A(t)\|_{\mathcal{Y}} \\ &\leq \underbrace{\left\|V(t) - \widetilde{V_A}(t)\right\|_{\mathcal{Y}}}_{(II)} + \underbrace{\|W_A(t)\|_{\mathcal{Y}}}_{(I)}. \end{aligned}$$

The proof is divided into three main steps : Step 0 is a preliminary analysis on the local Cauchy problem of the amplitude equation. Step 1 and 2 are the estimates of terms (I) and (II).

- **Step 0 : Cauchy Problem.** This part concerns the local Cauchy problem of the Ginzburg-Landau amplitude equation (8.4), throughout semi-group analysis. This analysis is performed in Section 8.4. The main results obtained in this step are the following.

Theorem 8.2

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $g(u) = u|u|^2$. For all $\varphi \in H^m(\mathbb{R})$ and $m \geq 1$, if $M > 0$ is such that $\|\varphi\|_{H^m} \leq M$, then there exists $T_M > 0$ and a unique solution $u \in C([0, T_M]; H^m(\mathbb{R}))$, with $\|u\|_{L^\infty(H^m)} \leq 2M$, to the Cauchy problem

$$\begin{cases} u(t,x) \in \mathbb{C}, \quad x \in \mathbb{R}, \\ \partial_t u = (c + id)u + (\nu + i\alpha)\Delta u + (a + ib)g(u), \quad \nu > 0, \\ u(0) = \varphi. \end{cases} \tag{8.11}$$

Corollary 8.3. *Let us consider $\varphi \in H^5(\mathbb{R})$. Then there exist $T_0 > 0$ and a solution u in $C([0, T_0], H^5(\mathbb{R}))$, to the problem (8.11). Moreover, we have*

$$u \in \bigcap_{j=0}^{2} C^j \left([0, T_0], H^{5-2j}(\mathbb{R})\right).$$

In other words, there exists a constant $C_A > 0$, such that for all k, ℓ in $\mathbb{N}$,

$$\boxed{\boldsymbol{k + 2\ell \leq 5 \Rightarrow \left\|\partial_x^k \partial_t^\ell u\right\|_{L^\infty([0,T_0],L^2)} \leq C_A.}} \tag{8.12}$$

- **Step 1 : estimate of $(\boldsymbol{I})$.** To estimate this term, we use essentially Corollary 8.3, resulting from Step 0.
  The term $W_A(t)$ corresponds to the order ε^3 in the formal approximation. But we want to estimate it in a time interval of length $\frac{1}{\varepsilon^2}$. We show that it does not blow up in this time-scale and that at the end we get an estimate of order $\varepsilon^{\frac{5}{2}}$. This is the object of Section 8.5. We obtain

  Proposition 8.4

  There exists a constant $C_W > 0$, independent of ε, such that for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$ it holds

  $$\boldsymbol{\|W_A(t)\|_{\mathcal{Y}} \leq C_W \varepsilon^{\frac{5}{2}}}.$$

- **Step 2 : estimate of $(\boldsymbol{II})$.** Section 8.6 is dedicated to this step. We want to estimate the error between the solution of the original problem and the formal approximation at order 3.
  In that aim, we introduce

  $$R(t) = \varepsilon^{-\alpha} \left\| V(t) - \widetilde{V_A}(t) \right\|_{\mathcal{Y}}, \tag{8.13}$$

  which means that we expect that $R(t)$ will stay bounded for a suitable power α. So the purpose is to find $\alpha > 1$ such that $R(t)$ remains bounded in timescale $\frac{1}{\varepsilon^2}$. To do this, we again proceed into three sub-steps.

  Step 2.1 We calculate the residual

  $$\rho_A = \dot{\widetilde{V}}_A - \mathscr{L}\widetilde{V}_A - \mathscr{N}(\widetilde{V}_A), \tag{8.14}$$

  which is what remains after substituting $\widetilde{V_A}$ into the equation (6.1). And then we find an estimate of its $\mathcal{Y}$-norm. We obtain

  Lemma 8.5. *There exists a constant $C_\rho > 0$ independent of ε, such that for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$,*

  $$\boxed{\boldsymbol{\|\rho_A(t)\|_{\mathcal{Y}} \leq C_\rho \varepsilon^{\frac{7}{2}}}.}$$

  Step 2.2 We calculate $\dot{R}(t)$ and obtain an ODE in which the non linearity depends on R, ρ_A and the difference $\mathscr{N}(V) - \mathscr{N}(\widetilde{V}_A)$. We deduce an integral formula for R given by

  Lemma 8.6. *$R(t)$ is solution to the integral equation*

  $$\begin{aligned} \boldsymbol{R(t)} &= \boldsymbol{S(t)R(0)} \\ &\quad + \int_0^t \boldsymbol{S(t-s)} \left\{ \underbrace{\boldsymbol{\varepsilon^{-\alpha}[\mathscr{N}(V) - \mathscr{N}(\widetilde{V_A})](s)}}_{(a)} - \underbrace{\boldsymbol{\varepsilon^{-\alpha}\rho_A(s)}}_{(b)} \right\} \boldsymbol{ds}, \end{aligned} \tag{8.15}$$

where $(S(t))_{t\geq 0}$ is the semi-group in the Hilbert $\mathcal{Y}$, generated by the bounded operator $\mathscr{L}$.

Step 2.3 Under an a priori estimate on $\|R(t)\|_{\mathcal{Y}}$ which reads : **there exists $\boldsymbol{D > 0}$ such that for all $\boldsymbol{t \in [0\,,\frac{\tau_0}{\varepsilon^2}]}$**,

$$\boxed{\|\boldsymbol{R(t)}\|_{\boldsymbol{\mathcal{Y}}} \leq \boldsymbol{D},} \tag{8.16}$$

we show that we can control the non linear term $\left\|\mathscr{N}(V) - \mathscr{N}(\widetilde{V}_A)\right\|_{\mathcal{Y}}$. We prove

Lemma 8.7. *There exists a constant $C_{\mathscr{N}}$ independent of ε such that for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$ we have*

$$\left\|\varepsilon^{-\alpha}[\mathscr{N}(V) - \mathscr{N}(\widetilde{V}_A))](t)\right\|_{\mathcal{Y}} \leq C_{\mathscr{N}}\varepsilon^2 \|R(t)\|_{\mathcal{Y}}.$$

Step 2.4 (end of the proof) Finally, combining all the estimates obtained in Step 2.1 and 2.3, and the integral formula obtained in step 2.2, we prove with the Gronwall Lemma, that $\alpha = \frac{3}{2}$ is a suitable power and that the following holds :

Proposition 8.8

There exists a constant $C_V > 0$ such that for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$, it holds

$$\left\|\boldsymbol{V(t) - \widetilde{V_A}(t)}\right\|_{\boldsymbol{\mathcal{Y}}} \leq \boldsymbol{C_V \varepsilon^{\frac{3}{2}}}.$$

To prove this estimate, that is, to prove that $R(t)$ stays bounded for $\varepsilon = \frac{3}{2}$, we use the fact that the semi-group $(S(t))_{t\geq 0}$ generated by $\mathscr{L}$ is quasi-bounded in the following way :

Proposition 8.9

There exists constants $\kappa > 0$ and $\sigma > 0$ independent of ε, such that we have

$$\text{for all } t \geq 0,\ \|S(t)\|_{\mathcal{Y}\to\mathcal{Y}} \leq \sigma e^{\kappa\varepsilon^2 t}.$$

The proof of this proposition is the object of Section 8.7.

8.3 Step 2.4 : end of the proof of Theorem 8.1

We admit all the results of Section 8.2 and use them now to prove Theorem 8.1. Thus we look for an estimate on $R(t)$. With the integral formula (8.15), we first write

$$\begin{aligned} \|R(t)\|_{\mathcal{Y}} &\leq \|S(t)R(0)\|_{\mathcal{Y}} \\ &+ \int_0^t \left\|S(t-s)\left\{\varepsilon^{-\alpha}[\mathscr{N}(V) - \mathscr{N}(\widetilde{V_A})](s) + \varepsilon^{-\alpha}\rho_A(s)\right\}\right\|_{\mathcal{Y}} ds. \end{aligned} \tag{8.17}$$

To go on, we need an estimate of the semi-group $S(t) = e^{t\mathscr{L}}$. We recall that $\mathscr{L}$ is a bounded operator and that the larger real parts in its spectrum are of order $O(\varepsilon^2)$ (see Proposition 6.9 in the spectral analysis). So there is no surprise in the fact that, as in finite dimension, the semi-group $e^{t\mathscr{L}}$ is exponentially bounded by $\sigma e^{\kappa\varepsilon^2 t}$. But here, we do not want the constant σ to depend on ε^2 and especially to be of order $O(\frac{1}{\varepsilon^2})$, as we would expect to at first look. We admit for the moment the result of Proposition 8.9 and conclude the proof of Theorem 8.1 with this semi-group estimate. Inequality (8.17) gives for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\begin{aligned}\|R(t)\|_{\mathcal{Y}} &\leq \sigma e^{\kappa\varepsilon^2 t}\,\|R(0)\|_{\mathcal{Y}} \\ &\quad +\sigma\int_0^t e^{\kappa\varepsilon^2 t}\left\{\left\|\varepsilon^{-\alpha}[\mathscr{N}(V)-\mathscr{N}(\widetilde{V_A})](s)\right\|_{\mathcal{Y}}+\|\varepsilon^{-\alpha}\rho_A(s)\|_{\mathcal{Y}}\right\}ds.\end{aligned}$$

Then, applying Lemma 8.5 and 8.7, we obtain

$$\|R(t)\|_{\mathcal{Y}} \leq \sigma e^{\kappa\tau_0}\,\|R(0)\|_{\mathcal{Y}}+\sigma e^{\kappa\tau_0}\left\{\int_0^t C_{\mathscr{N}}\varepsilon^2\,\|R(s)\|_{\mathcal{Y}}\,ds+\int_0^t \varepsilon^{-\alpha}C_\rho\varepsilon^{\frac{7}{2}}ds\right\}.$$

Moreover we have

$$\begin{aligned}\|R(0)\|_{\mathcal{Y}} &= \varepsilon^{-\alpha}\,\|V(0)-V_A(0)\|_{\mathcal{Y}}\,, \\ &\leq \varepsilon^{-\alpha}\left(\left\|V(0)-\widetilde{V_A}(0)\right\|_{\mathcal{Y}}+\|W_A(0)\|_{\mathcal{Y}}\right), \\ &\leq \varepsilon^{-\alpha}\left(d_0\varepsilon^{\alpha}+C_W\varepsilon^{\frac{5}{2}}\right),\end{aligned}$$

where the first term has been bounded by hypothesis on the initial condition (8.7) and the second one comes from Proposition 8.4 obtained in Step 1. Thus it follows for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$, under the a priori estimate (8.16), that

$$\|R(t)\|_{\mathcal{Y}} \leq \sigma e^{\kappa\tau_0}(d_0+c_W\varepsilon^{\frac{5}{2}}+C_\rho\varepsilon^{\frac{7}{2}-\alpha}t)+\sigma e^{\kappa\tau_0}C_{\mathscr{N}}\varepsilon^2\int_0^t\|R(s)\|_{\mathcal{Y}}\,ds.$$

We apply Gronwall's Lemma to $\Phi(t)=\|R(t)\|_{\mathcal{Y}}$. It comes

$$\begin{aligned}\|R(t)\|_{\mathcal{Y}} &\leq (d_0+C_W\varepsilon^{\frac{5}{2}}+C_\rho\varepsilon^{\frac{7}{2}-\alpha}t)e^{\sigma e^{\kappa\tau_0}C_{\mathscr{N}}\varepsilon^2 t}, \\ &\leq (2d_0+C_\rho\varepsilon^{\frac{7}{2}-\alpha}t)e^{c\varepsilon^2 t},\end{aligned}$$

for small ε and $c=\sigma C_{\mathscr{N}}e^{\kappa\tau_0}$. So we can at last discuss about the value of α. We must have $\alpha>1$ to have an error that is smaller than the amplitude A of the formal approximation, which is in $O(\varepsilon)$. And to conclude from the last inequality that $\|R(t)\|_{\mathcal{Y}}$ is bounded for $t\in[0\,,\frac{\tau_0}{\varepsilon^2}]$, we should have $\frac{7}{2}-\alpha\geq 2$, so that $\varepsilon^{\frac{7}{2}-\alpha}t$ does not blow up for t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$. So α must satisfy $\alpha\in\,]1\,,\frac{3}{2}]$. Thus we choose $\boldsymbol{\alpha=\frac{3}{2}}$ as stated in (8.7).

For this choice of α, it comes for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\begin{aligned} \|R(t)\|_{\mathcal{Y}} &\leq (2d_0 + C_\rho \varepsilon^2 t)e^{c\varepsilon^2 t}, \\ &\leq (2d_0 + C_\rho \tau_0)e^{c\tau_0}, \end{aligned} \tag{8.18}$$

and we can now choose $D = (2d_0 + C_\rho\tau_0)e^{c\tau_0}$. So that the a priori estimate (8.16) is satisfied in $[0\,,\frac{\tau_0}{\varepsilon^2}]$.

In conclusion, $R(t)$ stays bounded for t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$. And thus we have proved estimate (8.8) of Theorem 8.1.

□

8.4 Step 0 : Local Cauchy Problem for the complex Ginzburg-Landau equation (8.4)

In this section, our interest is the scalar complex Ginzburg-Landau PDE given by (8.4), which reads

$$\boxed{\partial_\tau A = (c + id)A + (\nu + i\alpha)\partial_{\xi\xi}A + (a + ib)|A|^2 A,}$$

with $a, b, c, d, \nu, \alpha \in \mathbb{R}$. One can find a general survey about the dynamics of solutions to the complex Ginzburg-Landau equation in [AK02]. Here, our concern is to prove Theorem 8.2, which says that we have local existence in time and that the regularity of the initial data, with respect to the space variable, propagates to the solutions.

Remark 8.10. *This result is similar to the one obtained for the Non Linear Schrödinger Equation (see [Caz03], H^m regularity for $m > d/2$, Theorem 4.10.1). In our case, the nonlinearity g is perfectly smooth. So the proof will be simpler than the proof of Theorem 4.10.1 in [Caz03], which bears on a general case.*

Proof : Let us first notice that if A is a solution of (8.11), then $u := e^{-c\tau}A$ is a solution to the equation with $c = 0$. As we want to prove only estimates in a $O(1)$-time scale, we can set $c = 0$. So we focus on the following equation

$$\partial_t u = idu + (\nu + i\alpha)\Delta u + (a + ib)|u|^2 u,$$

with $\nu > 0$ and $u = u(t, x) \in \mathbb{C}$.

The proof of Theorem 8.2 is then divided into three steps :

Step (i) We prove in subsection 8.4.1 that the linear operator is the infinitesimal generator of a contraction semi-group $G(t)$ in $X = L^2$.

Proposition 8.11

The operator $(T, D(T))$ given by Definition 8.13 is the infinitesimal generator of a *strongly continuous semi-group of contraction in $X = L^2(\mathbb{R})$, denoted by $(G(t))_{t\geq 0}$.*

Step (ii) We show in subsection 8.4.2 that $(G(t))_{t\geq 0}$ also defines a contraction semi-group in $H^m(\mathbb{R})$ for all $m \geq 1$.

Proposition 8.12

The restriction of the semi-group $(G(t))_{t\geq 0}$ to $H^m(\mathbb{R})$ for all $m \in \mathbb{R}$ again defines a contraction semi-group.

Step (iii) We write an equivalent integral formula and perform a Picard's fixed point Theorem and conclude the proof of 8.2 in subsection 8.4.3.

8.4.1 Step (i) : Semi-group theory

To begin, let us define the linear operator T and its domain.

Definition 8.13

Let define $D(T) = H^2(\mathbb{R})$ and $X = L^2(\mathbb{R}, \mathbb{C})$. Then for $u \in D(T)$, we define the operator T of dense domain $D(T)$ in X by

$$Tu = (\nu + i\alpha)\Delta u + idu,$$

with $\nu > 0$ and $\alpha, d \in \mathbb{R}$.

We then have

Proposition 8.14

For $\nu > 0$, T is a m-dissipative operator of dense domain.

Proof: We recall that T is m-dissipative if

mD1) $\forall\, u \in D(T), \forall \lambda > 0$, $\|u - \lambda T u\|_X \geq \|u\|_X$ and
mD2) $\forall\, \lambda > 0$ and $\forall\, f \in X$, $\exists\, u \in D(T)$ such that $u - \lambda T u = f$.

We consider in X the real scalar product $\langle u, v\rangle = \mathrm{Re}\,(\int_{\mathbb{R}} u\overline{v})$. With this scalar product, X is a real Hilbert space. Let us prove mD1) and mD2).

- mD1) is equivalent to $\langle Tu, u\rangle_X \leq 0$, $\forall u \in D(T)$. Here we have

$$\begin{aligned} \langle Tu, u\rangle_X &= \mathrm{Re}\left(id\|u\|_X^2 - (\nu + i\alpha)\|\nabla u\|_X^2\right), \\ &= -\nu\|\nabla u\|_X^2, \end{aligned}$$

  which is negative since $\nu > 0$.

- mD2) Let $\lambda \in (0, +\infty)$ and $f \in X$. We formulate the variationnal problem

$$\left\{ \begin{array}{l} u \in H^1(\mathbb{R}), \\ \langle u, v\rangle_X - \lambda\langle Tu, v\rangle_X = \langle f, v\rangle_X, \quad \forall v \in H^1(\mathbb{R}). \end{array} \right. \tag{8.19}$$

We define $b(u, v) = \langle u, v\rangle_X - \lambda\langle Tu, v\rangle_X = \int_{\mathbb{R}} u\overline{v} + \lambda\nu \int_{\mathbb{R}} \nabla u \nabla \overline{v}$. Let us check that b satisfies the hypotheses of the Lax-Milgram Theorem.

- if $u, v \in H^1(\mathbb{R})$ then

$$\begin{array}{rcl} |b(u, v)| & \leq & \|u\|_X \|v\|_X + \nu\lambda \|\nabla u\|_X \|\nabla v\|_X, \\ & \leq & (1 + \lambda v)\|u\|_{H^1} \|v\|_{H^1}. \end{array}$$

So b is a bilinear continuous map in $H^1(\mathbb{R})$.

- Let us prove that it is coercive. For $u \in H^1(\mathbb{R})$ then we have $b(u, u) = \|u\|_X^2 + \lambda\nu\|\nabla u\|_X^2$. Hence we deduce

$$b(u, u) \geq \min(1, \lambda\nu)\|u\|^2_{H^1}.$$

Thus it follows from the Lax-Milgram Theorem, that for all $f \in L^2$, there exists a unique $u \in H^1$ such that $b(u, v) = \langle f, v\rangle_X$. Hence this solution u is a solution in the sense of distributions to $u - \lambda Tu = f$. Moreover, since $u \in L^2$ and $f \in L^2$, it follows from this equation that $Tu \in L^2$ and then $\Delta u \in L^2$. So we have $u \in H^2 = D(T)$ and mD2) is satisfied : $(0, +\infty) \subset \rho(T)$. This concludes the proof of Proposition 8.14.

□

Then Proposition 8.11 comes as an immediate consequence of this proposition and of the Hille-Yosida Theorem (see for instance [CH90] Theorem 3.4.4).

8.4.2 Step (ii) : Proof of Proposition 8.12

The semi-group $(G(t))_{t\geq 0}$ is a contraction semi-group in H^m. It is well known that for $\varphi \in D(T)$, we have that $u(t) = G(t)\varphi$ is the unique solution of the Cauchy-problem (8.11) in $C([0, +\infty), D(T)) \cap C^1([0, +\infty); X)$ (see for instance [CH90], Theorem 3.1.1). To prove this proposition, let us introduce a representation formula for initial data in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Lemma 8.15. *For all $\varphi \in \mathcal{S}(\mathbb{R})$, we define*

$$u(t) = \mathcal{F}_I^{-1}\left[e^{-(\nu+i\alpha)\zeta^2 t + idt}\mathcal{F}_I[\varphi]\right],$$

where $\mathcal{F}_I$ denotes the Fourier integral $\mathcal{F}_I[u](\zeta) = \dfrac{1}{2\pi}\displaystyle\int_R e^{ix\zeta}u(x)dx$. Then we have $u(t) = G(t)\varphi$.

Proof of the lemma : we begin to prove that $u(t)$ defined by this formula is a solution to the Cauchy problem (8.11). The Schwartz space is stable by the Fourier transform, so let us now consider $\varphi \in \mathcal{S}(\mathbb{R})$. Then u is a solution of (8.11) if and only if

$$\begin{cases} \frac{d}{dt}\mathcal{F}_I[u] = -(\nu + i\alpha)\zeta^2 \mathcal{F}_I[u] + id\mathcal{F}_I[u], \\ \mathcal{F}_I[u](0) = \mathcal{F}_I[\varphi]. \end{cases}$$

The solution to this Cauchy problem is given by

$$\mathcal{F}_I[u](t,\zeta) = e^{\left(-(\nu+i\alpha)\zeta^2+id\right)t}\mathcal{F}_I[\varphi].$$

To conclude the proof of the lemma, it remains to prove that $u(t)$ is in $C([0,+\infty), H^2(\mathbb{R}))\cap C^1([0,+\infty); L^2(\mathbb{R}))$.
First, we notice that for $\varphi \in \mathcal{S}(\mathbb{R})$, we have $\mathcal{F}_I[u] = e^{-(\nu+i\alpha)\zeta^2 t + idt}\mathcal{F}_I[\varphi] \in \mathcal{S}(\mathbb{R})$, which is again equivalent to $u \in \mathcal{S}(\mathbb{R})$.
- $u \in C([0,+\infty), H^2(\mathbb{R}))$: let us prove that u is continuous at any t_0 in $[0,+\infty)$. We have

$$\begin{aligned} \|u(t) - u(t_0)\|_{H^2} &= \left\| (1+\zeta^2)\left(\mathcal{F}_I[u](t) - \mathcal{F}_I[u](t_0)\right)\right\|_{L^2_\zeta}, \\ &= \left\| (1+\zeta^2)\left(e^{p(\zeta)t} - e^{p(\zeta)t_0}\right)\mathcal{F}_I[\varphi]\right\|_{L^2_\zeta}, \end{aligned}$$

where we have denoted by $p(\zeta) = -(\nu + i\alpha)\zeta^2 + id$. Then it holds

$$\|u(t) - u(t_0)\|_{H^2} = \int_{\mathbb{R}} (1+\zeta^2)^2 \left| e^{p(\zeta)t} - e^{p(\zeta)t_0}\right|^2 |\mathcal{F}_I[\varphi]|^2 \, d\zeta.$$

Let us denote by $f(\zeta,t) = (1+\zeta^2)\left| e^{p(\zeta)t} - e^{p(\zeta)t_0}\right| |\mathcal{F}_I[\varphi]|$. We have

$$|f(\zeta,t)| \;\; \leq \;\; (1+\zeta^2)|\mathcal{F}_I[\varphi]|\left(\left|e^{p(\zeta)t}\right| + \left|e^{p(\zeta)t_0}\right|\right),$$

and given the sign of ν, it follows

$$|f(\zeta,t)|^2 \;\; \leq \;\; 4(1+\zeta^2)^2 \left|\mathcal{F}_I[\varphi]\right|^2.$$

The right hand side is independent of t and integrable since $\left|\mathcal{F}_I[\varphi]\right|^2 \in \mathcal{S}(\mathbb{R})$. Finally with $f(\zeta,t) \longrightarrow 0$ when $t \longrightarrow t_0$, we conclude by the dominated convergence Theorem that

$$\|u(t) - u(t_0)\|_{H^2} \longrightarrow 0 \quad \text{when} \quad t \longrightarrow t_0,$$

and thus $u \in C([0,+\infty), H^2(\mathbb{R}))$.

- $u \in C^1([0,+\infty), L^2(\mathbb{R}))$: we have

$$\left\| \frac{du}{dt}(t) - \frac{du}{dt}(t_0)\right\|_{L^2} \;\; = \;\; \int_{\mathbb{R}} |g(\zeta,t)|^2 \, d\zeta,$$

with $g(\zeta,t) = p(\zeta)f(\zeta,t)$. As p is a polynomial in ζ, we conclude exactly in the same way. Thus $u(t)$ is solution to the Cauchy problem (8.11) and lies in $C([0,+\infty),H^2(\mathbb{R}))\cap C^1([0,+\infty);L^2(\mathbb{R}))$. So by uniqueness, we conclude that $u(t)=G(t)\varphi, \forall t\geq 0$. And then the lemma is proved.

□

We now conclude the proof of Proposition 8.12, by using the density of the Schwartz space into $H^m(\mathbb{R})$. We know that $G(t):\mathcal{S}(\mathbb{R})\longrightarrow\mathcal{S}(\mathbb{R})$ is given by the explicit representation formula of the lemma. Moreover, we have

$$\begin{aligned}
\|u\|_{H^m} &= \left\|(1+\zeta^2)^{\frac{m}{2}}\mathcal{F}_I[u]\right\|_{L^2_\zeta},\\
&= \left\|(1+\zeta^2)^{\frac{m}{2}}e^{-(\nu+i\alpha)\zeta^2 t+idt}\mathcal{F}_I[\varphi]\right\|_{L^2_\zeta},\\
&\leq \left\|(1+\zeta^2)^{\frac{m}{2}}\mathcal{F}_I[\varphi]\right\|_{L^2_\zeta},
\end{aligned}$$

given the sign of ν. Hence we have $\|G(t)\|_{\mathcal{S}\longrightarrow H^m}\leq 1$. Applying finally the extension Theorem for linear continuous operator, we deduce that $G(t)$ can be extended to $H^m(\mathbb{R})$ with preservation of its norm. So it remains a contraction in H^m. Thus it concludes the proof of Proposition 8.12.

□

8.4.3 Step (iii) : Picard 's fixed point Theorem

We know that if

$$u\in L^\infty([0,T_0),H^m(\mathbb{R})),$$

then the two equations (8.11) and

$$u(t)=G(t)\varphi+\int_0^t G(t-s)g(u(s))ds, \tag{8.20}$$

have a sense in $H^{m-2}(\mathbb{R})$. Moreover, u satisfies (8.11) almost everywhere in $[0,T_0)$ if and only if, u satisfies (8.20) almost everywhere in $[0,T_0)$ (see [Caz03], Lemma 4.2.3).
We then introduce the integral operator $\mathscr{H}$ defined in H^m by

$$\mathscr{H}[u](t)=G(t)\varphi+\int_0^t G(t-s)g(u(s))ds. \tag{8.21}$$

Let us consider $\varphi\in H^m(\mathbb{R})$, with $m\geq 2$, $M>0$ such that $\|\varphi\|_{H^m}\leq M$ and $T_0>0$ to be chosen in the following. We define

$$B_K=\left\{u\in L^\infty([0,T_0],H^m(\mathbb{R}))\ \text{ such that }\ \|u\|_{L^\infty([0,T_0],H^m)}\leq K\right\}, \tag{8.22}$$

where $K>0$ will be precised later. We consider the following distance in B_K :

$$\text{for } u,v\in B_K,\quad d(u,v)=\|u-v\|_{L^\infty([0,T_0],H^m)}.$$

Then (B_K,d) is clearly a complete metric space. To apply the fixed point Theorem in (B_K,d), we have to prove the two points

(a) $\mathscr{H} : (B_K, d) \longrightarrow (B_K, d)$,

(b) $\mathscr{H}$ is a contraction.

Properties (a) and (b) rely on the following result.

Lemma 8.16.
$$\begin{aligned} F : (H^m(\mathbb{R}), \|.\|_{H^m}) &\longrightarrow (H^m(\mathbb{R}), \|.\|_{H^m}) \\ u &\longmapsto u|u|^2, \end{aligned}$$
is a Lipschitz function on the bounded subsets of $H^m(\mathbb{R})$.

Proof of the lemma : let $u, v \in H^m(\mathbb{R})$, $m \geq 2$, with $\|u\|_{H^m}, \|v\|_{H^m} \leq M$ and let us prove that there exists $C(M)$ a constant depending on M such that we have
$$\|F(u) - F(v)\|_{H^m} \leq C(M) \|u - v\|_{H^m}.$$
We have
$$\|F(u) - F(v)\|_{H^m}^2 = \sum_{k=0}^{m} \left\| \partial_x^k [F(u) - F(v)] \right\|_{L^2}^2.$$
So to begin, we compute $\partial_x^k[F(u) - F(v)]$. We denote by u_R and u_I the real and imaginary part of u. We then have
$$\begin{aligned} \partial_x^k[F(u)] &= \partial_x^k \left[u(u_R^2 + u_I^2) \right], \\ &= \sum_{j=0}^{k} \partial_x^{k-j} u \left[\partial_x^j(u_R^2) + \partial_x^j(u_I^2) \right], \\ &= \sum_{j=0}^{k} \sum_{\ell=0}^{j} \partial_x^{k-j} u \left(\partial_x^\ell u_R \partial_x^{j-\ell} u_R + \partial_x^\ell u_I \partial_x^{j-\ell} u_I \right). \end{aligned}$$
We deduce
$$\begin{aligned} \partial_x^k [F(u) - F(v)] &= \sum_{j=0}^{k} \sum_{\ell=0}^{j} \partial_x^{k-j} u \left(\partial_x^\ell u_R \partial_x^{j-\ell} u_R + \partial_x^\ell u_I \partial_x^{j-\ell} u_I \right) \\ &\qquad - \partial_x^{k-j} v \left(\partial_x^\ell v_R \partial_x^{j-\ell} v_R + \partial_x^\ell v_I \partial_x^{j-\ell} v_I \right). \end{aligned}$$
Let us factorize each term as follows for $j \in [\![0, k]\!]$ and $\ell \in [\![0, j]\!]$:
$$\begin{aligned} &\partial_x^{k-j} u \partial_x^\ell u_R \partial_x^{j-\ell} u_R - \partial_x^{k-j} v \partial_x^\ell v_R \partial_x^{j-\ell} v_R = \\ &\left(\partial_x^{k-j} u - \partial_x^{k-j} v \right) \partial_x^\ell u_R \partial_x^{j-\ell} u_R + \partial_x^{k-j} v \left(\partial_x^\ell u_R \partial^{j-\ell} u_R - \partial_x^\ell v_R \partial_x^{j-\ell} v_R \right) = \\ &\partial_x^{k-j} [u - v] \partial_x^\ell u_R \partial_x^{j-\ell} u_R + \partial_x^{k-j} v \partial_x^{j-\ell} u_R \partial_x^\ell [u_R - v_R] \\ &\qquad + \partial_x^\ell v_R \partial_x^{k-j} v \partial_x^{j-\ell} [u_R - v_R]. \end{aligned}$$
The same equality holds for u_I instead of u_R. So we see that $\partial_x^k [F(u) - F(v)]$ is a sum of terms of the form
$$\partial_x^{p_1} w_1 \partial_x^{p_2} w_2 \partial_x^{p_3} w_3, \tag{*}$$

where $p_1 + p_2 + p_3 = k$ and two of the w_i (say for instance w_1 and w_2) are equal to one of the functions v, u, u_R, v_R, u_I, v_I and the last one (w_3) is one of the differences $u - v$, $u_R - v_R$ or $u_I - v_I$. To estimate $\left\| \partial_x^k \left[F(u) - F(v) \right] \right\|_{L^2(\mathbb{R})}$, we use the well-known Sobolev injection $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$: there exists $C_{sob} > 0$ such that for all $w \in H^1(\mathbb{R})$, we have

$$\boldsymbol{\|w\|_{L^\infty} \leq C_{sob} \|w\|_{H^1}} \,.$$

For each term $(*)$ we then obtain an estimate of the form

$$\|\partial_x^{p_1} w_1 \partial_x^{p_2} w_2 \partial_x^{p_3} w_3\|_{L^2(\mathbb{R})} \leq \left\| \partial_x^{p_{i_1}} w_{i_1} \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x^{p_{i_2}} w_{i_2} \right\|_{L^\infty(\mathbb{R})} \left\| \partial_x^{p_{i_3}} w_{i_3} \right\|_{L^2(\mathbb{R})},$$

where $i_1, i_2, i_3 \in [\![1\,, 3]\!]$. We choose i_1, i_2, i_3 depending on the derivative powers p_1, p_2, p_3. Indeed,

- if $k \neq m$. Then $p_1 + p_2 + p_3 = k$ implies that all the p_i are different from m. And then we have $\partial_x^{p_i} w_i \in H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. We then choose to take the w_3 term in norm L^2 and w_1 and w_2 in norm L^∞. So we can write $i_3 = 3$ and

$$\begin{aligned} \|\partial_x^{p_1} w_1 \partial_x^{p_2} w_2 \partial_x^{p_3} w_3\|_{L^2(\mathbb{R})} &\leq \|\partial_x^{p_3} w_3\|_{L^2(\mathbb{R})} \prod_{i=1}^{2} \|\partial_x^{p_i} w_i\|_{L^\infty(\mathbb{R})}, \\ &\leq C_{sob}^2 \|\partial_x^{p_1} w_1\|_{H^1} \|\partial_x^{p_2} w_2\|_{H^1} \|\partial_x^{p_3} w_3\|_{L^2}. \end{aligned}$$

  Moreover, as $w_3 = u - v$, or $u_I - v_I$, or $u_R - v_R$, we have $\|\partial_x^{p_3} w_3\|_{L^2} \leq \|w_3\|_{H^m} \leq \|u - v\|_{H^m}$.
  In the same way, we have $\|\partial_x^{p_1} w_1\|_{H^1}, \|\partial_x^{p_2} w_2\|_{H^1} \leq \|u\|_{H^m}$ or $\|v\|_{H_m}$. Finally, as we have $\|u\|_{H^m}, \|v\|_{H^m} \leq M$, we conclude

$$\|\partial_x^{p_1} w_1 \partial_x^{p_2} w_2 \partial_x^{p_3} w_3\|_{L^2(\mathbb{R})} \leq C_{sob}^2 M^2 \|u - v\|_{H^m}.$$

- if $k = m$, then $p_1 + p_2 + p_3 = m$ and there exist configurations (p_1, p_2, p_3) for which one of the p_i is equal to m. For those configurations (say for example $p_1 = m$), we take the L^2-norm of this term ($i_3 = 1$), because $\partial_x^m w_1$ is in $L^2(\mathbb{R})$ but we cannot assert that it is in $L^\infty(\mathbb{R})$. If then $p_1 = m$ it ensures that $p_2, p_3 \leq m - 1$ and so $\partial_x^{p_2} w_2, \partial_x^{p_3} w_3 \in H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Thus in this case, we write

$$\begin{aligned} \|\partial_x^{p_1} w_1 \partial_x^{p_2} w_2 \partial_x^{p_3} w_3\|_{L^2(\mathbb{R})} &\leq \|\partial_x^{p_1} w_1\|_{L^2(\mathbb{R})} \prod_{i=2}^{3} \|\partial_x^{p_i} w_i\|_{L^\infty(\mathbb{R})}, \\ &\leq C_{sob}^2 \|w_1\|_{H^m} \|\partial_x^{p_2} w_2\|_{H^1} \|\partial_x^{p_3} w_3\|_{H^1} \\ &\leq C_{sob}^2 \|w_1\|_{H^m} \|w_2\|_{H^{p_2+1}} \|w_3\|_{H^{p_3+1}}, \\ &\leq C_{sob}^2 \|w_1\|_{H^m} \|w_2\|_{H^m} \|w_3\|_{H^m}, \\ &\leq C_{sob}^2 M^2 \|u - v\|_{H^m}. \end{aligned}$$

  Thus all the terms of the form $(*)$ are bounded by $C_{sob}^2 M^2 \|u - v\|_{H^m}$.

In conclusion, we have for all u, v in $H^m(\mathbb{R})$, such that $\|u\|_{H^m(\mathbb{R})}, \|v\|_{H^m(\mathbb{R})} \leq M$,

$$\|F(u) - F(v)\|_{H^m(\mathbb{R})} \leq C(M)\, \|u - v\|_{H^m(\mathbb{R})},$$

where $C(M) > 0$ is a constant depending on M. This concludes the proof of the lemma. We deduce immediately that $\boldsymbol{F \in C(H^m(\mathbb{R}), H^m(\mathbb{R}))}$ and the following lemma :

Lemma 8.17. *Let us consider* $u \in C([0,T_0], H^m(\mathbb{R}))$. *Then* $g(u) = |u|^2 u \in C\left([0,T_0], H^m(\mathbb{R})\right)$.

So we now prove that our integral operator $\mathscr{H}$, given by (8.21), satisfies the hypotheses of Picard's fixed point Theorem.

(a) $\mathscr{H} : B_K \longrightarrow B_K$. For $\varphi \in H^m(\mathbb{R})$, we have $G(t)\varphi \in H^m(\mathbb{R})$ and $t \longmapsto G(t)\varphi \in C([0,T_0], H^m(\mathbb{R}))$. Moreover, as from Lemma 8.17, we also have $g(u) \in C([0,T_0], H^m(\mathbb{R}))$, it follows that $\mathscr{H}[u] \in C([0,T_0], H^m(\mathbb{R}))$.

Let us now estimate $\|\mathscr{H}[u](t)\|_{H^m}$ for $t \in [0,T_0]$. Given (8.21), we have

$$\|\mathscr{H}[u](t)\|_{H^m} \leq \|G(t)\varphi\|_{H^m} + \int_0^t \|G(t-s)g(u(s))\|_{H^m}\, ds.$$

We deduce then by Proposition 8.12

$$\|\mathscr{H}[u](t)\|_{H^m} \leq \|\varphi\|_{H^m} + \int_0^t \|g(u(s))\|_{H^m}\, ds.$$

Moreover, as $u \in B_K$, we have $\|u(s)\|_{H^m} \leq K$ for all $s \in [0\,,T_0]$ and then by Lemma 8.16, $\|g(u(s))\|_{H^m} \leq C(K)\,\|u(s)\|_{H^m}$. Hence we have for all $t \in [0\,,T_0]$,

$$\begin{aligned}
\|\mathscr{H}[u](t)\|_{H^m} &\leq \|\varphi\|_{H^m} + C(K)\int_0^t \|u(s)\|_{H^m}\, ds,\\
&\leq \|\varphi\|_{H^m} + C(K)T_0\, \|u(s)\|_{L^\infty([0,T_0];H^m)}\, ds,\\
&\leq M + KC(K)T_0.
\end{aligned}$$

Thus, a sufficient condition to get (a) is

$$\boxed{M + KC(K)T_0 \leq K.} \tag{8.23}$$

(b) $\mathscr{H}$ is a contraction. We have for all t in $[0\,,T_0]$,

$$\begin{aligned}
\|\mathscr{H}[u](t) - \mathscr{H}[v](t)\|_{H^m} &= \left\|\int_0^t G(t-s)\left[g(u(s)) - g(v(s))\right] ds\right\|_{H^m},\\
&\leq \int_0^t \|g(u(s)) - g(v(s))\|_{H^m}\, ds,
\end{aligned}$$

and as $\|u\|_{L^\infty([0,T_0];H^m)}, \|v\|_{L^\infty([0,T_0];H^m)} \leq K$, it follows for all t in $[0\,,T_0]$ that

$$\begin{aligned}
\|\mathscr{H}[u](t) - \mathscr{H}[v](t)\|_{H^m} &\leq C(K)\int_0^t \|u(s) - v(s)\|_{H^m}\, ds,\\
&\leq C(K)T_0\, \|u - v\|_{L^\infty([0,T_0];H^m)}.
\end{aligned}$$

Thus, $\mathscr{H}$ is a contraction if the following condition is satisfied

$$\boxed{C(K)T_0 \leq \tfrac{1}{2}.} \tag{8.24}$$

We now choose K and T_0 to satisfy these two conditions. Given φ and then M such that $\boldsymbol{\|\varphi\|_{H^m} \leq M}$, we define $\boldsymbol{K = 2M}$. Next, we choose $T_0(M) = T_M > 0$ such that $\boldsymbol{C(2M)T_M \leq \frac{1}{2}}$ (for example $T_M = \frac{1/2}{1+C(2M)}$).

For these choices, conditions (8.23) and (8.24) are satisfied. And we deduce by the Picard's fixed point Theorem that there exists a unique solution $u \in C\left([0\,,T_M], H^m\right)$, with $\|u\|_{L^\infty([0,T_M];H^m)} \leq 2M$, to the integral equation (8.20). This concludes the proof of Theorem 8.2.

□

To conclude this preliminary study, we now set $m = 5$ and by a boot-strap argument, prove Corollary 8.3.

Proof: Let $u \in C([0\,,T_0], H^m(\mathbb{R}))$ given by Theorem 8.2. Then u satisfies the Ginzburg-Landau equation for all $t \in [0\,,T_0]$, i.e. we have

$$u_t = Tu + (a+ib)g(u),$$

with T given by Definition 8.13. This equality has to be taken in $H^3(\mathbb{R})$ because for $u(t) \in H^5(\mathbb{R})$, $Tu(t) \in H^3(\mathbb{R})$ and $g(u(t)) \in H^5(\mathbb{R})$. Thus we have

$$\begin{aligned} u \in C([0\,,T_0], H^5(\mathbb{R})) \quad &\Longrightarrow \quad u_t \in C([0\,,T_0], H^3(\mathbb{R})), \\ &\Longrightarrow \quad u \in C^1([0\,,T_0], H^3(\mathbb{R})). \end{aligned}$$

Iterating this once again, we obtain the result of Theorem 8.3. Estimate (8.12) is an immediate consequence of this.

□

8.4.4 Discussion about the signs of coefficients ν and c

The results of this section will be relevant only if the derived Ginzburg-Landau equation (8.4) satisfies the sign condition $\nu > 0$ required in Theorem 8.2. The coefficient ν is given by formula (8.5). This coefficient is a function of the critical parameter Λ^c. In addition to this sign condition, we recall that we chose $\boldsymbol{t}$ such that $\boldsymbol{t}.\nabla G(\Lambda^c) > 0$, in order to have stability for $\mu < 0$. And finally we want only simple roots to $P(X, q, 0)$, for all q. So the question is the following : do there exist $\Lambda^c \in \mathscr{R}^c$ and $\boldsymbol{t} = (a_{11}, b_{11}, \alpha_{11}, \beta_{11})$ in $\mathbb{R}^4$, such that we have

- $\boldsymbol{t}.\nabla G(\Lambda^c) > 0$,

- $\nu > 0$,
- $P(X, q, 0)$ has simple roots for all q in $[0, \pi]$?

This may not be possible for all Λ^c in $\mathscr{R}^c$. But our concern is to find at least one Λ, for which it is possible (and thus it will be true also in a neighbourhood of that Λ^c). Keeping our example $\Lambda^c = (-1, -1.5, -1, -1)$, these conditions are satisfied if we take t $= (1, 1, 0, 0)$, $(1, 1, -1, -1)$ or $(1, 1, 0, 1)$ for instance.
So there exist regions in $\mathscr{R}^c$ for which all the conditions are satisfied.

This concludes Step 0. We come now to the control of (I). In the following, we will consider $A(\tau, \xi)$ a solution to the Ginzburg-Landau amplitude equation (8.4) with an initial condition $A(0, .) \in H^5_\xi(\mathbb{R})$, defined in a certain time interval $[0\,, \tau_0]$. Then the regularity of A is given by Theorem 8.3. We also consider some Λ^c in $\mathscr{R}^c$ and a vector $\boldsymbol{t}$, such that the previous conditions are satisfied.

8.5 Step 1 : proof of Proposition 8.4

We recall that Step 1 is the estimate of $\|W_A(t)\|_{\mathcal{Y}}$, for $t \in [0\,, \frac{\tau_0}{\varepsilon^2}]$, where W_A is given by (8.9). We want to prove here the result given by Proposition 8.4. We have

$$\|W_A(t)\|^2_{\mathcal{Y}} = \|w_A(t)\|^2_{\ell^2(\mathbb{Z})} + \|\dot{w}_A(t)\|^2_{\ell^2(\mathbb{Z})} + \|\phi_A(t)\|^2_{\ell^2(\mathbb{Z})} \,.$$

Each of the terms w_A, $\dot{w}_A$ and ϕ_A is a priori of order ε^3. But we estimate it in a $O(\frac{1}{\varepsilon^2})$-long time interval, so it mays blow-up. The following technical lemma ensures that it is not the case.

Lemma 8.18. *For all $F \in H^1(\mathbb{R})$, all $\varepsilon \in (0, 1)$, and all $c \in \mathbb{R}$, we have*

$$\sum_{j \in \mathbb{Z}} \sup_{|s| \leq 1} |F(\varepsilon(j + s + c))|^2 \leq \frac{8}{\varepsilon} \|F\|^2_{H^1} \,.$$

This lemma can be found in [GM04], but as we make an extensively use of it, we give here its proof.
Proof: Let $x, y \in (j + c - 1, j + c + 1)$. We have by the integral formula :

$$F(x) - F(y) = \int_x^y F'(z)dz.$$

Then it follows

$$|F(x)| \leq |F(y)| + \int_{j+c-1}^{j+c+1} |F'(z)|dz.$$

Integrating this over y gives

$$\begin{aligned}
|F(x)|\int_{j+c-1}^{j+c+1} dy &\leq \int_{j+c-1}^{j+c+1} |F(y)|dy + \int_{j+c-1}^{j+c+1} |F'(z)|dz \int_{j+c-1}^{j+c+1} dy, \\
2|F(x)| &\leq \int_{j+c-1}^{j+c+1} \left(|F(y)| + 2|F'(y)|\right) dy, \\
|F(x)| &\leq \int_{j+c-1}^{j+c+1} \left(|F(y)| + |F'(y)|\right) dy.
\end{aligned}$$

With successively the inequality $a+b \leq \sqrt{2}\sqrt{a^2+b^2}$ and the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
|F(x)| &\leq \sqrt{2}\int_{j+c-1}^{j+c+1} \left(|F(y)|^2 + |F'(y)|^2\right)^{\frac{1}{2}} dy, \\
&\leq 2\left(\int_{j+c-1}^{j+c+1} \left(|F(y)|^2 + |F'(y)|^2\right) dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Then it follows

$$\begin{aligned}
\sup_{|s|\leq 1} |F(\varepsilon(j+c+s))|^2 &\leq 4\,\|F(\varepsilon .)\|^2_{H^1(j+c-1,j+c+1)}, \\
\sum_{j\in\mathbb{Z}} \sup_{|s|\leq 1} |F(\varepsilon(j+c+s))|^2 &\leq 8\,\|F(\varepsilon .)\|^2_{H^1(\mathbb{R})}.
\end{aligned}$$

And we conclude by computing $\|F(\varepsilon .)\|_{H^1}$ for $\varepsilon < 1$:

$$\begin{aligned}
\|F(\varepsilon .)\|^2_{H^1} &= \int_{\mathbb{R}} |F(\varepsilon x)|^2 dx + \int_{\mathbb{R}} |\varepsilon F(\varepsilon x)|^2 dx, \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}} |F(y)|^2 dy + \varepsilon^2 \int_{\mathbb{R}} |\partial_y F(y)|^2 \frac{dy}{\varepsilon}, \\
&= \frac{1}{\varepsilon}\left(\int_{\mathbb{R}} (|F(y)|^2 + \varepsilon^2 |\partial_y F(y)|^2) dy\right), \\
&\leq \frac{1}{\varepsilon}\,\|F\|_{H^1},
\end{aligned}$$

which concludes the proof of the lemma.

□

As a consequence of this lemma, we have : if $F \in H^1_\xi(\mathbb{R})$, then with $c = 0$ we obtain

$$\sum_{j\in\mathbb{Z}} |F(\varepsilon j)|^2 \leq \sum_{j\in\mathbb{Z}} \sup_{|s|\leq 1} |F(\varepsilon(j+s)|^2 \leq \frac{8}{\varepsilon}\,\|F\|^2_{H^1_\xi} < \infty.$$

And thus if $F \in H^1_\xi$, then $(F(\varepsilon j))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Let us now estimate $\|w_A(t)\|_{\ell^2(\mathbb{Z})}$, $\|\dot{w}_A(t)\|_{\ell^2(\mathbb{Z})}$ and $\|\phi_A(t)\|_{\ell^2(\mathbb{Z})}$.

- Estimate of $\|w_A(t)\|_{\ell^2(\mathbb{Z})}$. We have

$$w_A(t) = \varepsilon^3 \Omega'_{33} A^3(\tau, \varepsilon j) E^3(t) + c.c.$$

It follows

$$\|w_A(t)\|_{\ell^2(\mathbb{Z})} \leq 2|\Omega'_{33}| \left\| \left(A^3(\tau,\varepsilon j)\right)_j \right\|_{\ell^2(\mathbb{Z})}.$$

As $A(\tau,.) \in H^5_\xi(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we have $A^3(\tau,.) \in H^1_\xi(\mathbb{R})$. Indeed, for all $\tau \in [0\,,\tau_0]$,

$$\begin{aligned} \left\|A^3(\tau,.)\right\|^2_{H^1_\xi} &= \left\|A^3(\tau,.)\right\|^2_{L^2_\xi} + \left\|3A^2(\tau,.)\partial_\xi A(\tau,.)\right\|^2_{L^2_\xi}, \\ &\leq \left(\|A(\tau,.)\|^2_{L^\infty_\xi} \|A(\tau,.)\|_{L^2_\xi}\right)^2 + \left(3\,\|A(\tau,.)\|^2_{L^\infty_\xi} \|A(\tau,.)\|_{H^1}\right)^2, \\ &\leq 10\, C^4_{sob} C^6_A < \infty, \end{aligned}$$

where C_A is the constant coming from the estimates (8.12) and C_{sob} is coming from the Sobolev injection $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

We thus can use Lemma 8.18 to obtain

$$\left\| \left(A^3(\tau,\varepsilon j)\right)_j \right\|^2_{\ell^2(\mathbb{Z})} \leq \frac{8}{\varepsilon} \left\|A^3(\tau,.)\right\|_{H^1_\xi}.$$

And then it comes

$$\|w_A(t)\|_{\ell^2(\mathbb{Z})} \leq 2\sqrt{10}\sqrt{8}|\Omega'_{33}| C^2_{sob} C^3_A \varepsilon^{\frac{-1}{2}} \varepsilon^3.$$

So finally we have for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\boxed{\|\boldsymbol{w_A(t)}\|_{\boldsymbol{\ell^2}(\mathbb{Z})} \lesssim \boldsymbol{\varepsilon^{\frac{5}{2}}},} \tag{8.25}$$

where C is a positive constant.

In the following, many different constants will be denoted by C.

- Estimate of $\|\dot{w}_A(t)\|_{\ell^2(\mathbb{Z})}$. We recall that

$$\dot{w}_A(t) = 3i\omega A^3(\tau,\varepsilon j)\varepsilon^3 E^3(t) + 3\Omega'_{33}\partial_\tau A(\tau,\varepsilon j) A^2(\tau,\varepsilon j)\varepsilon^5 E^3(t) + c.c.$$

It implies

$$\|\dot{w}_A(t)\|_{\ell^2(\mathbb{Z})} \leq 6\omega\varepsilon^3 \left\|A^3(\tau,\varepsilon j)\right\|_{\ell^2(\mathbb{Z})} + 6\varepsilon^5 \left\|\partial_\tau A(\tau,\varepsilon j)A^2(\tau,\varepsilon j)\right\|_{\ell^2(\mathbb{Z})}.$$

We apply Lemma 8.18 with $F(\xi) = \partial_\tau A(\tau,\xi)A^2(\tau,\xi)$ for fixed τ in $[0\,,\tau_0]$. We have

$$\begin{aligned} \|F\|^2_{H^1_\xi} &= \|F\|^2_{L^2_\xi} + \|\partial_\xi F\|^2_{L^2_\xi} \\ &\leq C\Bigg(\left\|A^2(\tau,.)\right\|^4_{L^\infty_\xi} \|\partial_\tau A(\tau,.)\|^2_{L^2_\xi} + \|A(\tau,.)\|^4_{L^\infty_\xi} \|\partial_{\xi\tau} A(\tau,.)\|^2_{L^2_\xi} \\ &\qquad + \|A(\tau,.)\|^2_{L^\infty_\xi} \|\partial_\tau A(\tau,.)\|^2_{L^\infty_\xi} \|\partial_\xi A(\tau,.)\|^2_{L^2_\xi}\Bigg), \\ &\leq 3CC^4_{sob}C^6_A. \end{aligned}$$

thus by the lemma we deduce

$$\left\|\Big(\partial A(\tau,\varepsilon j)A^2(\tau,\varepsilon j)\Big)_j\right\|_{\ell^2(\mathbb{Z})} \lesssim \varepsilon^{\frac{1}{2}},$$

and finally we have

$$\|\dot{w}_A(t)\|_{\ell^2(\mathbb{Z})} \lesssim \varepsilon^{-\frac{1}{2}}(\varepsilon^3+\varepsilon^5),$$

thus for all $t\in[0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\boxed{\boldsymbol{\|\dot{w}_A(t)\|_{\ell^2(\mathbb{Z})} \lesssim \varepsilon^{\frac{5}{2}}.}} \tag{8.26}$$

- Estimate of $\|\phi_A(t)\|_{\ell^2(\mathbb{Z})}$. We recall

$$\phi_A(t)=\Omega_{31}\partial_\tau A(\tau,\varepsilon j)\varepsilon^3E(t)+\Omega_{33}A^3(\tau,\varepsilon j)\varepsilon^3E^3(t)+c.c.$$

Then it follows

$$\|\phi_A(t)\|_{\ell^2(\mathbb{Z})} \le C\varepsilon^3\left(\underbrace{\left\|(\partial_\tau A(\tau,\varepsilon j))_j\right\|_{\ell^2(\mathbb{Z})}}_{\lesssim\varepsilon^{\frac{1}{2}}}+\underbrace{\left\|\Big(A^3(\tau,\varepsilon j)\Big)_j\right\|_{\ell^2(\mathbb{Z})}}_{\lesssim\varepsilon^{\frac{1}{2}}}\right),$$

where we apply the lemma in this case with $F(\xi)=\partial_\tau A(\tau,\xi)\in H^1_\xi(\mathbb{R})$. We deduce for all $t\in[0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\boxed{\boldsymbol{\|\phi_A(t)\|_{\ell^2(\mathbb{Z})} \lesssim \varepsilon^{\frac{5}{2}}.}} \tag{8.27}$$

Combining the estimates (8.25), (8.26) and (8.27), we obtain that there exists $C_W>0$ such that for all $t\in[0\,,\frac{\tau_0}{\varepsilon^2}]$, we have

$$\boxed{\boldsymbol{\|W_A(t)\|_{\mathcal{Y}} \le C_W\varepsilon^{\frac{5}{2}}.}} \tag{8.28}$$

This concludes the first step and the proof of Proposition 8.4.

8.6 Step 2 : proof of Proposition 8.8

The purpose of Step 2 is to estimate $\left\|V(t)-\widetilde{V}_A(t)\right\|_{\mathcal{Y}}$, where V is the solution to the original system associated with an initial data close to the formal approximation $V_A(0)$ at time 0, and $\widetilde{V}_A$ is the formal approximation at order 3. Instead of assumption (8.7), we suppose first that $V_A(0)$ *close to* $V(0)$ means

$$\|V(0)-V_A(0)\|_{\mathcal{Y}} \le d_0\varepsilon^\alpha, \tag{8.29}$$

for a real power $\alpha > 1$, to determine. And we will see that the proof will lead us to take $\alpha = \frac{3}{2}$, as stated in (8.7).

As announced, we make the following ansatz for the error

$$\boxed{\boldsymbol{R(t) = \varepsilon^{-\alpha}(V(t) - \widetilde{V}_A(t)).}} \tag{8.30}$$

So we expect that $R(t)$ stays bounded for a well-chosen power α in the time interval $[0\,,\frac{\tau_0}{\varepsilon}]$. To do this, we follow substeps 2.1-2.3.

Step 2.1 : estimate of the residual ρ_A (proof of Lemma 8.5) We now compute the residual ρ_A defined by (8.14). We have

$$\rho_A = \begin{pmatrix} 0 \\ \rho_A^2 \\ \rho_A^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \ddot{\widetilde{v}}_A - \widetilde{v}_A - a_1\dot{\widetilde{v}}_A - b_1\widetilde{\psi}_A - \ell^2\Delta_d\,\widetilde{v}_A - N(\dot{\widetilde{v}}_A, \widetilde{\psi}_A) \\ \dot{\widetilde{\psi}}_A - \alpha_1\dot{\widetilde{v}}_A - \beta_1\widetilde{\psi}_A \end{pmatrix} \in \mathcal{Y}.$$

In chapter 7, we have made a formal derivation at order 3. So in the computation of the residual ρ_A it will remain only terms of order ε^4 and more. We obtain :

- Estimate of $\left\|\rho_A^3(t)\right\|_{\ell^2(\mathbb{Z})}$:

$$\begin{aligned} \rho_{A,j}^3(t) \quad &= \quad (3\Omega_{33} - 3\alpha_1\Omega_{33}')A^2(\tau,\varepsilon j)\partial_\tau A(\tau,\varepsilon j)\varepsilon^5 E^3(t) \\ &\qquad +\Omega_{31}\partial_{\tau\tau}A(\tau,\varepsilon j)\varepsilon^5 E(t) + c.c. \end{aligned}$$

Then it comes

$$\begin{aligned} |\rho_{A,j}^3| \quad &\leq \quad C\varepsilon^5\left(|\partial_{\tau\tau}A(\tau,\varepsilon j)| + |A^2(\tau,\varepsilon j)||\partial_\tau A(\tau,\varepsilon j)|\right), \\ \sum_{j\in\mathbb{Z}}|\rho_{A,j}^3|^2 \quad &\leq \quad C\varepsilon^{10}\left(\underbrace{\sum_{j\in\mathbb{Z}}|\partial_{\tau\tau}A(\tau,\varepsilon j)|^2}_{\lesssim\varepsilon^{-1}} + \|A(\tau,.)\|_{L_\xi^\infty}^4\underbrace{\sum_{j\in\mathbb{Z}}|\partial_\tau A(\tau,\varepsilon j)|^2}_{\lesssim\varepsilon^{-1}}\right), \end{aligned}$$

where we applied twice Lemma 8.18 with $F(\xi) = \partial_{\tau\tau}A(\tau,\xi)$ and $F(\xi) = \partial_\tau A(\tau,\xi) \in H^1(\mathbb{R})$. So we have for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\boxed{\left\|\boldsymbol{\rho_A^3(t)}\right\|_{\boldsymbol{\ell^2(\mathbb{Z})}} \lesssim \boldsymbol{\varepsilon^{\frac{9}{2}}}.} \tag{8.31}$$

- Estimate of $\left\|\rho_A^2(t)\right\|_{\ell^2(\mathbb{Z})}$. We have

$$\rho_A^2(t) = \ddot{\widetilde{v_A}} + \widetilde{v_A} - a_1\dot{\widetilde{v_A}} - b_1\widetilde{\psi_A} - \ell^2\Delta_d\,\widetilde{v_A} - a_3\dot{\widetilde{v_A}}^3 - b_3\widetilde{\psi_A}^3 - N_1(\dot{\widetilde{v_A}}) - N_2(\widetilde{\psi_A}).$$

We recall that we have $\widetilde{V}_A(t) = \begin{pmatrix} \widetilde{v}_A \\ \dot{\widetilde{v}}_A \\ \widetilde{\psi}_A \end{pmatrix}$, with

$$\begin{aligned} \widetilde{v}_a^j(t) &= A(\tau, \varepsilon j)\varepsilon E(t) + \Omega_{33}' A^3(\tau, \varepsilon j)\varepsilon^3 E^3(t) + c.c., \\ \widetilde{\psi}_A^j(t) &= \Omega_{11} A(\tau, \varepsilon j)\varepsilon E(t) + \Omega_{31}\partial_\tau A(\tau, \varepsilon j)\varepsilon^3 E(t) \\ &\qquad + \Omega_{33} A^3(\tau, \varepsilon j)\varepsilon^3 E^3(t) + c.c. \end{aligned}$$

So substituting $\widetilde{v}_A$ and $\widetilde{\psi}_A$ gives

$$\begin{aligned} \rho_{A,j}^2(t) &= \varepsilon^5 \left[E(t) \left\{ \partial_{\tau\tau} A - r_1 \ell^2/4! + (3a_3\omega^2 - 3b_3\Omega_{11}^2 \overline{\Omega_{31}}) A^2 \partial_\tau \overline{A} \right.\right. \\ &\qquad \left. + (9a_3 i\omega^3 \Omega_{33}' - 3b_3 \overline{\Omega_{11}}^2 \Omega_{33}) A|A|^4 \right\} \\ &\quad + E^3(t) \left\{ (18i\omega\Omega_{33}' + a_3\omega^2 - 3a_1\Omega_{33}') A^2 \partial_\tau A - \ell^2 \Omega_{33}' \partial_{\xi\xi} A^3 \right\} \\ &\quad \left. + E^5(t) \left\{ 9a_3 i\omega^3 \Omega_{33}' - 3b_3 \Omega_{11}^2 \Omega_{33} \right\} A^5 \right] \\ &\quad + h.o.t. \\ &\quad - N_1(\widetilde{\dot{v}_A}) - N_2(\widetilde{\psi_A}), \end{aligned}$$

where

$$r_1 = \partial_\xi^4 A(\tau, \xi + \varepsilon\theta_1^+) - \partial_\xi^4 A(\tau, \xi - \varepsilon\theta_1^-), \quad \theta_1 \in (0, 1),$$

and $h.o.t$ are terms of of the form

$$\varepsilon^k E^n \left\{ \text{sum of terms of the form } \left[\overline{A}^p A^q (\partial_\tau A)^m (\partial_\tau \overline{A})^\ell, \partial_{\xi\xi} A, r_1 \right] \right\},$$

with $k \geq 9$. We recall that in the cubic case, we have $\widehat{N_1}(\widetilde{\dot{v}_{A,j}}) = O(\widetilde{\dot{v}_{A,j}}^4)$ and $N_2(\widetilde{\psi_A}) = O(\widetilde{\psi_A}^4)$. We will see that these terms are the predominant ones. Indeed, Theorem 8.3 ensures that all these terms are in $H_\xi^1(\mathbb{R})$. Thus for each of them, we apply Lemma 8.18 with $c = 0$ or $c = \theta_i^\pm$ (in the case of terms like $\partial_\xi^4 A(\tau, \varepsilon j \pm \varepsilon\theta)$) and obtain that the ℓ^2-norm is in $O(\frac{1}{\sqrt{\varepsilon}})$. So all of those terms give an estimate in order $\frac{\varepsilon^5}{\sqrt{\varepsilon}} = \varepsilon^{\frac{9}{2}}$.

Remark 8.19. *We notice that to have $\partial_{\tau\tau} A \in H^1(\mathbb{R})$ and $\partial_\xi^4 A \in H_\xi^1(\mathbb{R})$ we need $H_\xi^5(\mathbb{R})$-regularity for the initial condition $A(0,.)$. Hence our choice of regularity in the theorem.*

It remains to estimate the $\widehat{N_1}$ and $\widehat{N_2}$ terms. We have

$$\begin{aligned} \widetilde{\dot{v}_{A,j}}(t) &= i\omega\varepsilon A(\tau, \varepsilon j)E(t) + \varepsilon^3 \partial_\tau A(\tau, \varepsilon j)E(t) + 3i\omega\Omega_{33}'\varepsilon^3 A^3(\tau, \varepsilon j)E^3(t) \\ &\quad + 3\Omega_{33}'\varepsilon^5 \partial_\tau A(\tau, \varepsilon j) A^2(\tau, \varepsilon j)E^3(t) + c.c. \end{aligned}$$

We then deduce

$$\begin{aligned} |\widehat{N_1}(\widetilde{\dot{v}_{A,j}}(t))| &\leq C|\widetilde{\dot{v}_{A,j}}(t)|^4, \\ &\leq C\varepsilon^4 \left(\sum_{m_1+\ldots+m_4=4} \left| A^{m_1} (\varepsilon^2 \partial_\tau A)^{m_2} (\varepsilon^2 A^3)^{m_3} (\varepsilon^4 A^2 \partial_\tau A)^{m_4} \right| \right), \\ &\leq C\varepsilon^4 \left(\sum_{m_1+\ldots+m_4=4} \left| A^{m_1+3m_3+2m_4} (\partial_\tau A)^{m_2+m_4} \right| \right). \end{aligned}$$

We then use Lemma 8.18 to estimate the ℓ^2-norm of each term of the form $|A^m\partial_\tau A^{m'}|$ and each of them is in $O(\frac{1}{\sqrt{\varepsilon}})$. So finally we obtain a bound in $\frac{\varepsilon^4}{\sqrt{\varepsilon}} = \varepsilon^{\frac{7}{2}}$ for $\left\|\widehat{N_1}(\dot{\widetilde{v_A}}(t))\right\|_{\ell^2(\mathbb{Z})}$. Given (8.10),we proceed exactly in the same manner for the estimate of $\left\|\widehat{N_2}(\widetilde{\psi_A}(t))\right\|_{\ell^2(\mathbb{Z})}$, and obtain a bound in $\varepsilon^{\frac{7}{2}}$. Thus, combining all the previous estimates, we obtain for all t in $[0\,,\frac{\tau_0}{\varepsilon^2}]$,

$$\boxed{\left\|\boldsymbol{\rho_A^2(t)}\right\|_{\boldsymbol{\ell^2}(\mathbb{Z})} \lesssim \boldsymbol{\varepsilon^{\frac{7}{2}}}.} \tag{8.32}$$

Finally with estimates (8.32) and (8.31) we deduce Lemma 8.5.

Step 2.2 : integral formula for $\boldsymbol{R(t)}$ (proof of Lemma 8.6) We then compute the derivative of $R(t) = \varepsilon^{-\alpha}\left(V(t) - \widetilde{V_A}(t)\right)$. We have

$$\begin{aligned} \dot{R}(t) &= \varepsilon^{-\alpha}\left(\dot{V}(t) - \dot{\widetilde{V_A}}(t)\right), \\ &= \varepsilon^{-\alpha}\left(\mathscr{L}(V - \widetilde{V_A}) + \mathscr{N}(V) - \mathscr{N}(\widetilde{V_A}) - \rho_A\right). \end{aligned}$$

Thus, R satisfies the differential equation in the Banach space $\mathcal{Y}$,

$$\dot{R}(t) = \mathscr{L}R(t) + \varepsilon^{-\alpha}[\mathscr{N}(V) - \mathscr{N}(\widetilde{V_A})] - \varepsilon^{-\alpha}\rho_A.$$

Equivalently, $R(t)$ is solution to the integral equation (8.15) given by Lemma 8.6. We recall that our purpose is to find a value of $\alpha > 1$ such that $\|R(t)\|_{\mathcal{Y}}$ is bounded in the time interval $[0\,,\frac{\tau_0}{\varepsilon^2}]$. We already have an estimate of the $\mathcal{Y}$-norm of the (b) term. We now look for an estimate of the $\mathcal{Y}$-norm of (a).

Step 2.3 : proof of Lemma 8.7 We recall that

$$\mathscr{N}(V) = \begin{pmatrix} 0 \\ N(V) \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} v \\ \dot{v} \\ \psi \end{pmatrix} \in \mathcal{Y},$$

where

$$N\,(V) = (N_1(\dot{v}_j) + N_2(\psi_j))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

We then denote by

$$\varepsilon^{-\alpha}[\mathscr{N}(V) - \mathscr{N}(\widetilde{V_A})] = {}^T(0, M(t), 0) \in \mathcal{Y},$$

where $M(t) = (M_j(t))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, with

$$\begin{aligned} &M_j(t) = M_j^1(t) + M_j^2(t), \\ &M_j^1(t) = \varepsilon^{-\alpha}[N_1(\dot{v}_j) - N_1(\dot{\widetilde{v_{A,j}}})], \\ &M_j^2(t) = \varepsilon^{-\alpha}[N_2(\psi_j) - N_2(\widetilde{\psi_{A,j}})]. \end{aligned}$$

We also denote by $R(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix} \in \mathcal{Y}$. We then write

$$\begin{aligned}
\varepsilon^\alpha M_j^1 &= N_1\left(\varepsilon^\alpha r_{2,j} + \dot{v_{A,j}}\right) - N_1\left(\dot{\widetilde{v_{A,j}}}\right), \quad j = 1,2, \\
M_j^1 &= r_j^2 \widehat{N_1}'(\dot{\widetilde{v_{A,j}}} + \varepsilon^\alpha r_j^2 \theta_j^1), \quad j = 1,2,
\end{aligned}$$

with $\theta_j^1 \in (0,1)$. The same holds for M_j^2 :

$$M_j^2 = r_j^3 N_2'(\dot{\widetilde{\psi_{A,j}}} + \varepsilon^\alpha r_j^3 \theta_j^2), \quad \theta_j^2 \in (0,1), \quad j = 1,2.$$

We know that *in the cubic case*, $N_1'(x) = O(x^2)$ and $N_2'(y) = O(y^2)$. Let us call $\boldsymbol{x_j = \dot{\widetilde{v_{A,j}}} + \varepsilon^\alpha r_j^2 \theta_j^1}$ and $\boldsymbol{y_j = \dot{\widetilde{\psi_{A,j}}} + \varepsilon^\alpha r_j^3 \theta_j^2}$. We are going to prove now that x_j and y_j are of order ε if t is in $[0\,,\frac{\tau_0}{\varepsilon^2}]$. And thus, $N_1'(x_j)$ and $N_2'(y_j)$ will be of order ε^2 for $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$.

For this purpose, we make first an a priori estimate on $R(t)$: **there exists $\boldsymbol{D > 0}$ such that for all $\boldsymbol{t}$ in $\boldsymbol{[0\,,\frac{\tau_0}{\varepsilon^2}]}$, we have**

$$\boxed{\|\boldsymbol{R(t)}\|_{\mathcal{Y}} \leq \boldsymbol{D.}} \tag{8.33}$$

The bound D will be chosen in the following. Given (8.10), we have

$$\begin{aligned}
x_j &= i\omega A\varepsilon E + \partial_\tau A \varepsilon^3 E + 3i\omega\Omega'_{33} A^3 \varepsilon^3 E^3 \\
&\quad + 3\Omega'_{33} A^2 \partial_\tau A \varepsilon^5 E^3 + c.c + \theta_j^1 \varepsilon^\alpha r_j^2(t).
\end{aligned}$$

Thus it comes

$$\begin{aligned}
|x_j(t)| &\leq C\Big(\varepsilon^3 \|\partial_\tau A\|_{L^\infty_\xi} + \varepsilon \|A\|_{L^\infty_\xi} + \varepsilon^3 \|A\|^3_{L^\infty_\xi} \\
&\qquad + \varepsilon^5 \|A\|^2_{L^\infty_\xi} \|\partial_\tau A\|_{L^\infty_\xi} + \varepsilon^\alpha |r_j^2(t)|\Big), \\
&\leq C\left(\varepsilon C' + \varepsilon^\alpha |r_j^2(t)|\right),
\end{aligned}$$

where C' is a constant depending on C_{sob} and the constant C_A is coming from (8.12). Now we use the a priori estimate (8.16). For all $j \in \mathbb{Z}$, we have

$$\begin{aligned}
|x_j(t)| &\leq C(\varepsilon + \varepsilon^\alpha \|R(t)\|_{\mathcal{Y}}), \\
|x_j(t)| &\leq C(\varepsilon + \varepsilon^\alpha D).
\end{aligned}$$

So under the a priori estimate and since $\alpha > 1$, we deduce that there exists $C_1 > 0$ such that for all $j \in \mathbb{Z}$, for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$, we have

$$|x_j(t)| \leq C_1 \varepsilon.$$

We prove the same for y_j : there exists $C_2 > 0$ such that for all $j \in \mathbb{Z}$, for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$, we have

$$|y_j(t)| \leq C_2\varepsilon.$$

We deduce from these two estimates and from ${N_1}'(x) = O(x^2)$ and ${N_2}'(y) = O(y^2)$ the following ones, which hold for small ε : there exists $L_1 > 0$ and $L_2 > 0$ such that for all $j \in \mathbb{Z}$, for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$, we have

$$\begin{array}{rcl} |{N_1}'(x_j(t))| & \leq & L_1\varepsilon^2, \\ |{N_2}'(y_j(t))| & \leq & L_2\varepsilon^2. \end{array}$$

Thus it implies

$$\begin{array}{rcl} \left\|M^1(t)\right\|_{\ell^2(\mathbb{Z})} & = & \left\|r_j^2(t){N_1}'(x_j(t))\right\|_{\ell^2(\mathbb{Z})}, \\ & \leq & L_1\varepsilon^2\left\|r_j^2(t)\right\|_{\ell^2(\mathbb{Z})} \leq L_1\varepsilon^2\left\|R(t)\right\|_{\ell^2(\mathbb{Z})}, \\ \left\|M^2(t)\right\|_{\ell^2(\mathbb{Z})} & \leq & L_2\varepsilon^2\left\|R(t)\right\|_{\ell^2(\mathbb{Z})}. \end{array}$$

And finally we obtain the estimate of Lemma 8.7 since $\left\|\varepsilon^{-\alpha}[\mathscr{N}(V) - \mathscr{N}(\widetilde{V}_A))]\right\|_{\mathcal{Y}} = \left\|M(t)\right\|_{\ell^2(\mathbb{Z})}$.

Remark 8.20. *What would happened if we supposed that the nonlinearity is quadratic ? Then, we would have had ${N_{1,2}}'(x) = O(x)$ and at the end of Step 2.3, we would have had for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$,*

$$\begin{array}{rcl} |{N_1}'(x_j(t))| & \leq & L_1\varepsilon, \\ |{N_2}'(y_j(t))| & \leq & L_2\varepsilon, \end{array}$$

instead of estimates in ε^2. And consequently, the bound in Lemma 8.7 would have been $C_{\mathscr{N}}\varepsilon\left\|R(t)\right\|_{\mathcal{Y}}$ instead of $C_{\mathscr{N}}\varepsilon^2\left\|R(t)\right\|_{\mathcal{Y}}$. Thus estimate (8.18) *of $R(t)$ would have been replaced by : for all $t \in [0\,,\frac{\tau_0}{\varepsilon^2}]$,*

$$\left\|R(t)\right\|_{\mathcal{Y}} \leq (2d_0 + C_\rho\varepsilon^2 t)e^{c\varepsilon t},$$

and we see that we would have not been able to conclude because the exponential term blows up. So the time scaling $\tau = \varepsilon^2 t$ is not appropriate for quadratic nonlinearities. In the quadratic case, we should change this time scaling. But in this case, other difficulties can arise. For instance in nonlinear optics, the quadratic interaction is responsible for the rectification effectss (the quadratic interaction produces a non oscillating mean mode, see [Lan98]).

8.7 Proof of Proposition 8.9

We recall that the expression of the linear bounded operator $\mathscr{L}$ in $\mathcal{Y} = \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ is

$$\mathscr{L} = \begin{pmatrix} 0 & 1 & 0 \\ -1 - \ell^2 \Delta_d & a_1 & b_1 \\ 0 & \alpha_1 & \beta_1 \end{pmatrix}.$$

We want to prove that the semi-group $S(t) = e^{t\mathscr{L}}$, $t \geq 0$ is quasi-bounded of type $\kappa\varepsilon^2$, and with constants independent of ε. The steps of the proof are the following :

- **Step (I)**. We write $e^{t\mathscr{L}}$ as the inverse Laplace transform of the resolvent $R(\zeta, \mathscr{L})$.
- **Step (II)**. Since we have no explicit formula for $R(\zeta, \mathscr{L})$ in $\mathcal{Y}$, but as we have one for its symbol $\check{R}(\zeta, q, \mathscr{L})$ in $\mathcal{Z} = L^2_q(T) \times L^2_q(T) \times L^2_q(T)$ (formula (6.6)), we prove that the Fourier transform operation commutes with $\boldsymbol{\mathcal{F}}$, in order to work in the $\mathcal{Z}$ space with $\check{R}(\zeta, q, \mathscr{L})$.
- **Step (III)**. We estimate the inverse Laplace integral in the space $\mathcal{Z}$ with the help of explicit formula (6.6).

Remark 8.21. *Here if we had a Hopf bifurcation, we would just have to project onto a finite dimension space and then proceed easily as in finite dimension for this part of the spectrum. But here we cannot separate the spectrum. So to tackle to this problem, we have to look at the singularities of the resolvent.*

Remark 8.22. *The Hille Yosida Theorem gives also an estimate for $e^{t\mathscr{L}}$. But the hypotheses of this theorem are too difficult to obtain. Indeed, we have to prove estimates of type $\|R(\zeta, \mathscr{L})\| \leq M(\zeta - \beta)^{-k}$, for all $\zeta > \beta$, where β is such that $[\beta\, , +\infty[\, \subset \rho(\mathscr{L})$ and for all k in $\mathbb{N}^*$.*

• **Step (I)**. Let us prove the following lemma.

Lemma 8.23. *Let Γ a simple closed curve surrounding the spectrum of $\mathscr{L}$. Then we have for all $t \geq 0$,*

$$e^{t\mathscr{L}} = \frac{1}{2\pi i} \int_\Gamma e^{\zeta t} R(\zeta, \mathscr{L}) d\zeta.$$

Remark 8.24. *This lemma says that the semi-group $e^{t\mathscr{L}}$ is the inverse Laplace transform of the resolvent $R(\zeta, \mathscr{L})$.*

<u>*Proof:*</u> To prove this lemma, we can first try to prove that
$U(t) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma e^{\zeta t} R(\zeta, \mathscr{L}) d\zeta$ satisfies all the properties of the continuous semi-group $e^{t\mathscr{L}}$. But for this, we need an estimate of the semi-group at infinity. This is done for example in [Ioo72]. It is supposed that the spectrum is contained in a sector $|\arg \zeta| \leq \frac{\pi}{2} + \omega$ and that outside of a sector $|\arg \zeta| \leq \frac{\pi}{2} + \omega - \varepsilon$, we have an estimate of type $R(\zeta, \mathscr{L}) \leq \frac{M_\varepsilon}{|\zeta|}$.

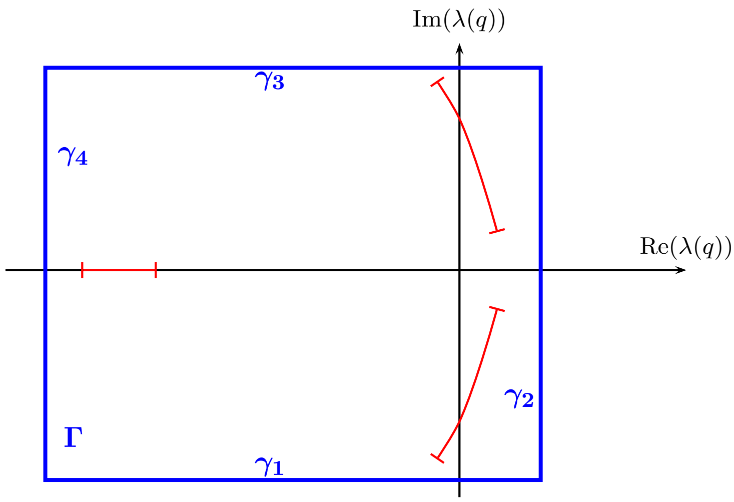


FIGURE 8.1 : Spectrum of $\mathscr{L}$ and closed curve Γ.

Here to prove the formula, we use a more general result, Theorem 11.3.1, stated in [HP74]. The proof of this Theorem is more difficult. But it gives that for any *bounded* operator $\mathscr{L}$, we indeed have the inverse Laplace formula of Lemma 8.23 without assuming any estimate on the resolvent.

$\square$

• **Step (II)**. We use now the results of Section 6. We already know the shape of the spectrum (see figure 6.2) and Proposition 6.9 says that the larger real part is in $O(\varepsilon^2)$. We know use these two pieces of information and the explicit expression of $\check{R}$ to estimate the semi-group.

To begin, let us prove that it is equivalent to work in $\mathcal{Z}$ and consider $\check{R}(\zeta, q, \mathscr{L})$ instead of working in $\mathcal{Y}$ with $R(\zeta, \mathscr{L})$. Let $\boldsymbol{\mathcal{F}^{-1}}$ be the inverse Fourier transform described at section 6. We have

Lemma 8.25. *For all $F \in \mathcal{Y}$, it holds*

$$\boldsymbol{\mathcal{F}^{-1}}\left[\frac{1}{2\pi i}\int_\Gamma e^{t\zeta} R(\zeta, \mathscr{L}) F d\zeta\right] = \frac{1}{2\pi i}\int_\Gamma e^{t\zeta} \boldsymbol{\mathcal{F}^{-1}}\left[R(\zeta, \mathscr{L}).F\right] d\zeta.$$

<u>*Proof:*</u> The operator $\boldsymbol{\mathcal{F}^{-1}} : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \longrightarrow L^2(T) \times L^2(T) \times L^2(T)$ is a continuous one-to-one map. We now transform the complex integral into a sum of real integrals by considering a parametrization of the curve $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where each γ_j is a segment (see figure 8.1). We introduce $\zeta_j : [0, 1] \longrightarrow \gamma_j$ a continuous parametrization

of γ_j. We then have

$$\frac{1}{2\pi i}\int_\Gamma e^{t\zeta}R(\zeta,\mathscr{L})Fd\zeta=\sum_{j=1}^{4}c_j\int_0^1 e^{t\zeta_j(x)}R(\zeta_j(x),\mathscr{L}).Fdx,$$

where c_j is a constant $c_j=\gamma_j'(x)$. We denote by $f_p(x)=e^{t\zeta_j(x)}R(\zeta_j(x),\mathscr{L}).F$ for $p\in\llbracket 1\,,4\rrbracket$. It is well-known that the resolvent $R(.,\mathscr{L})$ is holomorphic in the resolvent set $\rho(\mathscr{L})$. Thus, all the f_p are continuous functions from $[0\,,1]$ into the Hilbert space $\mathcal{Y}$. So the question is the following : do we have

$$\boldsymbol{\mathcal{F}^{-1}}\left[\int_0^1 g(x)dx\right]=\int_0^1\boldsymbol{\mathcal{F}^{-1}}[g(x)]dx,$$

for a continuous function g in $[0,1]$, taking values in the Banach ? The answer is yes and it is a classical result for Bochner integrals (using Riemann sums). Hence we have the result for each real integrals f_p.

$\square$

It then comes from this lemma that for all $t\geq 0$, and for all $F\in\mathcal{Y}$,

$$\boxed{\boldsymbol{\mathcal{F}^{-1}\left[e^{t\mathscr{L}}.F\right]=\frac{1}{2\pi i}\int_\Gamma e^{t\zeta}\check{R}(\zeta,q,\mathscr{L}).\check{F}d\zeta.}}\tag{8.34}$$

We want to estimate the $\mathcal{Y}$-norm of $e^{t\mathscr{L}}.F$. As the operator $\mathcal{F}^{-1}$ is an isometry, it comes from (8.34) that

$$\boldsymbol{\left\|e^{t\mathscr{L}}.F\right\|_{\mathcal{Y}}=\left\|\mathcal{F}^{-1}\left[e^{t\mathscr{L}}.F\right]\right\|_{\mathcal{Z}}=\left\|\frac{1}{2\pi i}\int_\Gamma e^{t\zeta}\check{R}(\zeta,q,\mathscr{L}).\check{F}d\zeta\right\|_{\mathcal{Z}}.}$$

• **Step (III).**
Finally, we want to estimate the $\mathcal{Z}$-norm of $\frac{1}{2\pi i}\int_\Gamma e^{t\zeta}\check{R}(\zeta,q,\mathscr{L}).\check{F}d\zeta$, where $\check{R}$ is given by explicit formula (6.6). For this, we compute each component of the integral operator $\frac{1}{2\pi i}\int_\Gamma e^{t\zeta}\check{R}(\zeta,q,\mathscr{L})d\zeta$ in $\mathcal{Z}$,

$$I_{kl}(q)=\frac{1}{2\pi i}\int_\Gamma e^{t\zeta}\check{r}_{kl}(\zeta,q)d\zeta,\quad k,l\in\llbracket 1\,,3\rrbracket.$$

We denote by $\check{R}(\zeta,q,\mathscr{L}).\check{F}=\check{V}$ with $\check{F}={}^T(\check{f}_1,\check{f}_2,\check{f}_3)$ and $\check{V}={}^T(\check{u}_1,\check{v}_2,\check{v}_3)$. We then have for $j=1,2,3$,

$$\check{v}_j(q)=I_{j1}(q)\check{f}_1(q)+I_{j2}(q)\check{f}_2(q)+I_{j3}(q)\check{f}_3(q).$$

(1) We have $I_{11}=\dfrac{1}{2\pi i}\displaystyle\int_\Gamma e^{t\zeta}\check{r}_{11}(\zeta,q)d\zeta$, where

$$\check{r}_{11}(q)=\frac{b_1\alpha_1-(\beta_1-\zeta)(a_1-\zeta)}{P(\zeta,q,\varepsilon^2)},$$

with $P(\zeta, q, \varepsilon^2)$ given by (6.5). The singularities of $\check{r}_{11}$ are the 3 roots of $P(\zeta, q, \varepsilon^2)$: $\zeta^+(q, \varepsilon^2) = \overline{\zeta^-(q, \varepsilon^2)} \in \mathbb{C}$ and $\zeta^R(q, \varepsilon^2) \in \mathbb{R}^-$ which are inside Γ. Let us compute the residue of $e^{t\zeta}\check{r}_{11}$ at these points. As by the assumptions made on Λ^c, we have 3 simple roots, the residues are

$$Res(e^{t\zeta}\check{r}_{11}; \zeta^\sigma(q, \varepsilon^2)) = \frac{b_1\alpha_1 - (\beta_1 - \zeta^\sigma(q, \varepsilon^2))(a_1 - \zeta^\sigma(q, \varepsilon^2))}{\partial_\zeta P(\zeta^\sigma(q, \varepsilon^2), q)} e^{t\zeta^\sigma(q,\varepsilon^2)},$$

where $\sigma \in \{\pm, R\}$. Moreover, we have for $\sigma_0 \in \{\pm, R\}$,

$$\partial_\zeta P(\zeta^{\sigma_0}(q, \varepsilon^2), q, \varepsilon^2) = \prod_{\sigma \in \{\pm, R\} \setminus \sigma_0} \left(\zeta^{\sigma_0}(q, \varepsilon^2) - \zeta^\sigma(q, \varepsilon^2) \right).$$

Thus we deduce by the Cauchy's residue Theorem that

$$\begin{aligned} I_{11}(q) &= \frac{b_1\alpha_1 - (\beta_1 - \zeta^+)(a_1 - \zeta^+)}{(\zeta^+ - \zeta^-)(\zeta^+ - \zeta^R)} e^{t\zeta^+} + c.c. \\ &\quad + \frac{b_1\alpha_1 - (\beta_1 - \zeta^R)(a_1 - \zeta^R)}{(\zeta^R - \zeta^-)(\zeta^R - \zeta^+)} e^{t\zeta^R}. \end{aligned}$$

In the same manner, we find

(2)

$$\begin{aligned} I_{12}(q) &= \frac{\beta_1 - \zeta^+}{(\zeta^+ - \zeta^-)(\zeta^+ - \zeta^R)} e^{t\zeta^+} + c.c \\ &\quad + \frac{\beta_1 - \zeta^R}{(\zeta^R - \zeta^-)(\zeta^R - \zeta^+)} e^{t\zeta^R}. \end{aligned}$$

(3)

$$\begin{aligned} I_{13}(q) &= \frac{-b_1}{(\zeta^+ - \zeta^-)(\zeta^+ - \zeta^R)} e^{t\zeta^+} + c.c \\ &\quad + \frac{-b_1}{(\zeta^R - \zeta^-)(\zeta^R - \zeta^+)} e^{t\zeta^R}. \end{aligned}$$

We can now estimate $\|\check{v}_1\|_{L^2_q}$:

$$\begin{aligned} \|\check{v}_1\|^2_{L^2_q} &= \int_0^{2\pi} \left| I_{11}(q)\check{f}_1(q) + I_{12}(q)\check{f}_2(q) + I_{13}(q)\check{f}_3(q) \right|^2 dq, \\ &\le 4 \left(\int_0^{2\pi} \left| I_{11}(q)\check{f}_1(q) \right|^2 + \int_0^{2\pi} \left| I_{21}(q)\check{f}_2(q) \right|^2 + \int_0^{2\pi} \left| I_{13}(q)\check{f}_3(q) \right|^2 \right). \end{aligned}$$

We have

$$\begin{aligned} \left| I_{11}(q)\check{f}_1(q) \right|^2 &\le 4 \left(2 \frac{|b_1\alpha_1 - (\beta_1 - \zeta^+)(a_1 - \zeta^+)|^2}{|\zeta^+ - \zeta^-|^2 |\zeta^+ - \zeta^R|^2} e^{2\mathrm{Re}\,\zeta^+ t} \right. \\ &\quad \left. + \frac{|b_1\alpha_1 - (\beta_1 - \zeta^R)(a_1 - \zeta^R)|^2}{|\zeta^R - \zeta^-|^2 |\zeta^R - \zeta^+|^2} e^{2\mathrm{Re}\,\zeta^R t} \right) |\check{f}_1(q)|^2. \end{aligned} \tag{8.35}$$

It follows from Proposition 6.9 that we have

$$\begin{aligned}&\Big(\zeta^+(q,\varepsilon^2)-\zeta^-(q,\varepsilon^2)\Big)\Big(\zeta^+(q,\varepsilon^2)-\zeta^R(q,\varepsilon^2)\Big)=2i\,\mathrm{Im}[\zeta^+(q,\varepsilon^2)]\Big(\zeta^+(q,\varepsilon^2)-\zeta^R(q,\varepsilon^2)\Big)\\&=2i\left[\mathrm{Im}\zeta^+(q,0)+O(\varepsilon^2)\right]\left[\zeta^+(q,0)-\zeta^R(q,0)+O(\varepsilon^2)\right].\end{aligned}$$

Since there is no collision in the spectrum, it follows that $\mathrm{Im}\zeta^+(q,0)\neq 0$ and $\zeta^+(q,0)-\zeta^R(q,0)\neq 0$ for all $q\in[0\,,2\pi]$. Thus for q describing the compact interval $[0\,,2\pi]$, the continuous function

$$q\longrightarrow\Big(\zeta^+(q,\varepsilon^2)-\zeta^-(q,\varepsilon^2)\Big)\Big(\zeta^+(q,\varepsilon^2)-\zeta^R(q,\varepsilon^2)\Big)$$

does not vanish. Thus its inverse is bounded. For small ε, it holds : there exists $C_+>0$ such that for all $q\in[0\,,\pi]$

$$\frac{1}{\left|(\zeta^+(q)-\zeta^-(q))\,(\zeta^+(q)-\zeta^R(q))\right|^2}\leq C_+.$$

The same holds for $\partial_\zeta P(\zeta^{-,R},q,\varepsilon^2)$: there exist $C_->0$, and $C_R>0$ such that for all $q\in[0\,,\pi]$

$$\begin{aligned}\frac{1}{\left|(\zeta^-(q)-\zeta^+(q))\,(\zeta^-(q)-\zeta^R(q))\right|^2}&\leq C_-,\\ \frac{1}{\left|(\zeta^R(q)-\zeta^+(q))\,(\zeta^R(q)-\zeta^-(q))\right|^2}&\leq C_R.\end{aligned}$$

On the other side, the numerator $q\mapsto|b_1\alpha_1-(\beta_1-\zeta^\sigma)(a_1-\zeta^\sigma)|^2\left|e^{2t\mathrm{Re}\,\zeta^\sigma}\right|$, for $\sigma\in\{\pm,R\}$, is continuous in the compact $[0\,,2\pi]$ and thus bounded. So there exists $K_\sigma>0$ such that for all $q\in[0\,,2\pi]$, for all $t\geq 0$, we also have

$$|b_1\alpha_1-(\beta_1-\zeta^\sigma)(a_1-\zeta^\sigma)|^2e^{2t\mathrm{Re}\,\zeta^\sigma}\leq K_\sigma,.$$

Then we deduce from (8.35) that

$$\begin{aligned}\left|I_{11}(q)\check{f}_1(q)\right|^2&\leq 4C_+(K_+e^{2t\,\mathbf{Re}[\zeta^+(q,\varepsilon^2)]}+K_-e^{2t\,\mathbf{Re}[\zeta^-(q,\varepsilon^2)]}+K_R)\left|\check{f}_1(q)\right|^2,\\&\leq 4C_+(K_+e^{2t\,\mathbf{Re}[\zeta^+(0,\varepsilon^2)]}+K_-e^{2t\,\mathbf{Re}[\zeta^-(0,\varepsilon^2)]}+K_R)\left|\check{f}_1(q)\right|^2,\\&\leq K_1e^{2\kappa\varepsilon^2t}\left|\check{f}_1\right|^2,\end{aligned}$$

since $q=0$ corresponds to the larger real part, and since from Proposition 6.9, we have : there exists $\kappa>0$ such that $\mathbf{Re}[\zeta^+(0,\varepsilon^2)]\leq\kappa\varepsilon^2$.

The same arguments hold for the estimates of $\left|I_{12}(q)\check{f}_2(q)\right|^2$ and $\left|I_{13}(q)\check{f}_3(q)\right|^2$, and thus we have also that there exist $K_2>0$ and $K_1>0$ such that for all $t\geq 0$

$$\begin{aligned}\left|I_{12}(q)\check{f}_2(q)\right|^2&\leq K_2e^{2\kappa\varepsilon^2t}\left|\check{f}_2(q)\right|^2,\\ \left|I_{13}(q)\check{f}_3(q)\right|^2&\leq K_3e^{2\kappa\varepsilon^2t}\left|\check{f}_3(q)\right|^2.\end{aligned}$$

Finally combining those inequalities, we obtain that there exists $C_1 > 0$ and $\kappa > 0$ such that for all $t \geq 0$

$$\boxed{\boldsymbol{\|\check{v}_1\|_{L^2_q} \leq C_1 \left\|\check{F}\right\|_{\mathcal{Z}} e^{\kappa\varepsilon^2 t}.}}$$

(4) $I_{21} = \displaystyle\int_\Gamma e^{t\zeta} r_{21}(\zeta, q) d\zeta$, with $r_{21}(\zeta, q) = 1 + \zeta r_{11}(\zeta, q)$. Thus it implies that $I_{21} = \displaystyle\int_\Gamma e^{t\zeta}\zeta r_{11}(\zeta, q) d\zeta$. So we just replace r_{11} by ζr_{11} in (1), which does not change anything because the maps $q \mapsto \zeta^\sigma(q, \varepsilon^2)$ are continuous and bounded in $[0\,, 2\pi]$.

(5) and (6) : Similarly, we have $I_{22} = \displaystyle\int_\Gamma e^{t\zeta} r_{22}(\zeta, q) d\zeta$ and $I_{23} = \displaystyle\int_\Gamma e^{t\zeta} r_{23}(\zeta, q) d\zeta$, with $r_{22}(\zeta, q) = \zeta r_{12}(\zeta, q)$ and $r_{23}(\zeta, q) = \zeta r_{13}(\zeta, q)$. Thus in these three cases, we conclude as for (1)-(3) : there exists $C_2 > 0$ such that for all $t \geq 0$

$$\boxed{\boldsymbol{\|\check{v}_2\|_{L^2_q} \leq C_2 \left\|\check{F}\right\|_{\mathcal{Z}} e^{\kappa\varepsilon^2 t}.}}$$

(7), (8) and (9) : Here we have $r_{31} = -\alpha_1 \dfrac{1 + \zeta r_{11}}{\beta_1 - \zeta}$, $r_{32} = -\alpha_1 \dfrac{\zeta r_{12}}{\beta_1 - \zeta}$, $r_{33} = \dfrac{1 - \alpha_1 \zeta r_{13}}{\beta_1 - \zeta}$. These functions have no singularity in β_1, since β_1 is in the resolvent set of $\mathscr{L}$. So in these cases, we again have three residues, corresponding to the singularities $\zeta^{\pm, R}(q, \varepsilon^2)$. And the residues have also bounded numerator and denominator. Thus we conclude again that there exists $C_3 > 0$ such that for all $t \geq 0$,

$$\boxed{\boldsymbol{\|\check{v}_3\|_{L^2_q} \leq C_3 \left\|\check{F}\right\|_{\mathcal{Z}} e^{\kappa\varepsilon^2 t}.}}$$

Hence, for our choice of Λ^c and particularly by the fact that $P(X, q, \mu$ has no double roots, we have that there exists $\sigma > 0$ and $\kappa > 0$ such that for all $\check{F} \in \mathcal{Z}$, for all $t \geq 0$,

$$\left\|\check{R}(\zeta, ., \varepsilon^2)\check{F}\right\|_{\mathcal{Z}} \leq \sigma e^{\kappa\varepsilon^2 t} \left\|\check{F}\right\|_{\mathcal{Z}},$$

or equivalently, there exists $\sigma > 0$ and $\kappa > 0$ such that for all $t \geq 0$, for all $\zeta \in \rho(\mathscr{L})$,

$$\boldsymbol{\left\|R(\zeta, \varepsilon^2)\right\|_{\mathcal{Y} \to \mathcal{Y}} \leq \sigma e^{\kappa\varepsilon^2 t}.}$$

This concludes the proof of Proposition 8.9.

□

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